A New Class Of Meromorphic Multivalent Functions Involving Certain Linear Operator*

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Abstract

Making use of certain extended derivative operator of Ruscheweyh type, we introduce a new class \( J_\mu(\lambda, \mu, \alpha) \) of meromorphic multivalent function in the punctured disk \( D = \{ z \in \mathbb{C}, 0 < |z| < 1 \} \), and obtain some sufficient conditions for the functions belonging to this class.

Keywords and Phrases: Meromorphic multivalent functions; Meromorphic starlike function; Meromorphic convex function; Meromorphic close-to-convex function; Hadamard product (or convolutions), Ruscheweyh derivative.

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1. Introduction and Definitions

Let $\Sigma(p)$ denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (p \in \mathbb{N}), \quad (1.1)$$

which are analytic and $p$-valent in the punctured unit disk

$$\mathbb{D} = \{ z : z \in \mathbb{C}, \, 0 < |z| < 1 \}.$$ 

We denote by $\Sigma^*(p, \alpha), \Sigma_k(p, \alpha)$ and $\Sigma_c(p, \alpha)$, the subclasses of the class $\Sigma(p)$, which are defined (for $0 \leq \alpha < p$, $p \in \mathbb{N}$) as follows:

$$\Sigma^*(p, \alpha) = \left\{ f : f \in \Sigma(p), \, \Re \left( -\frac{zf'(z)}{f(z)} \right) > \alpha \right\} \quad (z \in \mathbb{D}), \quad (1.2)$$

$$\Sigma_k(p, \alpha) = \left\{ f : f \in \Sigma(p), \, \Re \left( -1 - \frac{zf''(z)}{f'(z)} \right) > \alpha \right\} \quad (z \in \mathbb{D}), \quad (1.3)$$

and

$$\Sigma_c(p, \alpha) = \left\{ f : f \in \Sigma(p), \, \Re \left( -\frac{f'(z)}{z^{-p-1}} \right) > \alpha \right\} \quad (z \in \mathbb{D}), \quad (1.4)$$

Note that $\Sigma^*(p, \alpha), \Sigma_k(p, \alpha)$ and $\Sigma_c(p, \alpha)$, are the well known subclasses of $\Sigma(p)$ consisting of meromorphic multivalent functions which are respectively starlike, convex and close-to-convex functions of order $\alpha(0 \leq \alpha < p)$. Furthermore $\Sigma^*(1, \alpha) = \Sigma^*(\alpha), \Sigma_k(1, \alpha) = \Sigma_k(\alpha)$ and $\Sigma_c(1, \alpha) = \Sigma_c(\alpha)$, where $\Sigma^*(\alpha), \Sigma_k(\alpha)$ and $\Sigma_c(\alpha)$ are subclasses of $\Sigma(1)$ consisting meromorphic univalent functions which are respectively starlike, convex and close-to-convex of order $\alpha(0 \leq \alpha < 1)$. We refer to Liu and Srivastava [2], Mogra [4], Raina and Srivastava [5] and Xu and Yang [8] for related work on the subject of meromorphic functions.

For $f(z) \in \Sigma(p)$ given by (1.1) and $g(z) \in \Sigma(p)$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \quad (p \in \mathbb{N}), \quad (1.5)$$
the Hadamard product (or convolution) of \( f \) and \( g \) is defined by
\[
(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (f * g)(z).
\tag{1.6}
\]

The extended linear derivative operator of Ruscheweyh type for the functions belonging to the class \( \Sigma(p) \)
\[
D_{\ast}^{\lambda,p} : \Sigma(p) \to \Sigma(p),
\]
is defined by the following convolution:
\[
D_{\ast}^{\lambda,p} f(z) = \frac{1}{z^p(1-z)^{\lambda+1}} * f(z) \quad (\lambda > -1; f \in \Sigma(p)). \tag{1.7}
\]

In terms of binomial coefficients, (1.7) can be written as
\[
D_{\ast}^{\lambda,p} f(z) = z^{-p} + \sum_{k=1}^{\infty} \binom{\lambda + k}{k} a_k z^{k-p} \quad (\lambda > -1; f \in \Sigma(p)). \tag{1.8}
\]

In particular when \( \lambda = n \) \((n \in \mathbb{N})\), it is easily observed from (1.7) and (1.8) that
\[
D_{\ast}^{n,p} f(z) = \frac{z^{-p} (z^{n+p} f(z))^{(n)}}{n!} \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.9}
\]

The definition (1.7) of linear operator \( D_{\ast}^{\lambda,p} \) is motivated essentially by familiar Ruscheweyh operator \( D^{\lambda} \), which has been used widely on the space of analytic and univalent functions (see for details, [7]). A linear operator \( D^{\lambda,p} \) analogous to \( D_{\ast}^{\lambda,p} \) (defined by (1.7)), was considered recently by Raina and Srivastava [6] on the space of analytic and \( p \)-valent functions in \( U \) \((U = \mathbb{D} \cup \{0\})\).

We remark in passing that a more general convolution operator then the operator \( D_{\ast}^{\lambda,p} \) considered recently by Liu and Srivastava [2].

By using the operator \( D_{\ast}^{\lambda,p} (\lambda > -p; \ p \in \mathbb{N}) \) given by (1.7), we now introduce a new class of meromorphically \( p \)-valent analytic functions defined as follows:

**Definition 1.** A function \( f(z) \in \Sigma(p) \), is said to be a member of the class \( J_{\lambda}(\mu, \alpha) \) if and only if
\[
\left| \frac{z^{p+1}}{z^p D_{\ast}^{\lambda,p} f(z)} \left( D_{\ast}^{\lambda,p} f(z) \right)' \right|^{\mu-1} + p < p - \alpha \tag{1.10}
\]
Note that condition (1.10) implies that
\[ \Re \left( -\frac{z^{p+1} \left( D_{\ast}^{\lambda,p} f(z) \right)'}{\left( z^p D_{\ast}^{\lambda,p} f(z) \right)^{\mu-1}} \right) > \alpha. \] (1.11)

It is obvious that
\[ J_p(0, 2, \alpha) = \Sigma^*(p, \alpha) \quad \text{and} \quad J_p(0, 1, \alpha) = \Sigma_c(p, \alpha). \]

The object in the present paper is to obtain some sufficient conditions of functions belonging to the above defined subclass \( J_p(\lambda, \mu, \alpha) \).

2. Main Results

In our present investigation of the function class \( J_p(\lambda, \mu, \alpha) \) \( (\lambda > -1; \mu \geq 0; 0 \leq \alpha < p) \), we shall require the following Lammss.

Lemma 1 (see, [1]). Let the nonconstant function \( w(z) \) be analytic in \( \mathbb{U} \), with \( w(0) = 0 \). If \( |w(z)| \) attains its maximum value on the circle \( |z| = r < 1 \) at a point \( z_0 \in \mathbb{U} \), then
\[ z_0 w'(z_0) = kw(z_0), \]
where \( k \geq 1 \) is a real number.

Lemma 2 (see, [3]). Let \( S \) be a set in the complex plane \( \mathbb{C} \) and suppose that \( \phi(z) \) is a mapping from \( \mathbb{C}^2 \times \mathbb{U} \) to \( \mathbb{C} \) which satisfies \( \Phi(ix, y; z) \notin S \) for all \( z \in \mathbb{U} \), and for all real \( x, y \) such that \( y \leq -(1 + x^2)/2 \). If the function \( q(z) = 1 + q_1 z + q_2 z^2 + \cdots \) is analytic in \( \mathbb{U} \) such that \( \phi(q(z), zq'(z); z) \in S \) for all \( z \in \mathbb{U} \), then \( \Re (q(z)) > 0 \).

Making use of Lemma 1, we first prove

Theorem 1. Let \( p \in \mathbb{N} \), \( \gamma \geq 0 \), \( \lambda > -p \), \( \mu \geq 0 \) and \( 0 \leq \alpha < p \). If \( f(z) \in \Sigma(p) \) satisfies the following inequality
\[ \left| 1 + p + \frac{z \left( D_{\ast}^{\lambda,p} f(z) \right)''}{\left( D_{\ast}^{\lambda,p} f(z) \right)'} - (\mu - 1) \left( p + \frac{z \left( D_{\ast}^{\lambda,p} f(z) \right)''}{D_{\ast}^{\lambda,p} f(z)} \right) - \gamma \left( \frac{z^{p+1} \left( D_{\ast}^{\lambda,p} f(z) \right)'}{\left( z^p D_{\ast}^{\lambda,p} f(z) \right)^{\mu-1} + p} \right) \right| \]
< \frac{(p - \alpha)(2p - \alpha) + 1}{2p - \alpha}, \quad (2.1)

then \( f(z) \in J_\mu(\lambda, \mu, \alpha) \).

**Proof.** Define the function \( w(z) \) by

\[
\frac{z^{p+1} \left( D_\lambda^{\lambda,p} f(z) \right)'}{\left( z^p D_\lambda^{\lambda,p} f(z) \right)^{\mu-1}} = -p + (\alpha - p)w(z), \quad (2.2)
\]

then \( w(z) \) is analytic in \( U \) and \( w(0) = 0 \). Differentiating logarithmically both sides of (2.2) with respect to \( z \), we get

\[
p+1 + \frac{z \left( D_\lambda^{\lambda,p} f(z) \right)''}{\left( D_\lambda^{\lambda,p} f(z) \right)'} - (\mu - 1) \left( p + \frac{z \left( D_\lambda^{\lambda,p} f(z) \right)'}{D_\lambda^{\lambda,p} f(z)} \right) = \frac{(p - \alpha)zw'(z)}{p + (p - \alpha)w(z)}. \quad (2.3)
\]

Now using (2.2) in (2.3), we find that

\[
p+1 + \frac{z \left( D_\lambda^{\lambda,p} f(z) \right)''}{\left( D_\lambda^{\lambda,p} f(z) \right)'} - (\mu - 1) \left( p + \frac{z \left( D_\lambda^{\lambda,p} f(z) \right)'}{D_\lambda^{\lambda,p} f(z)} \right) - \gamma \left( \frac{z^{p+1} \left( D_\lambda^{\lambda,p} f(z) \right)'}{\left( z^p D_\lambda^{\lambda,p} f(z) \right)^{\mu-1}} + p \right) = \gamma(p - \alpha)w(z) + \frac{(p - \alpha)zw'(z)}{p + (p - \alpha)w(z)}. \quad (2.4)
\]

Let us suppose that there exist \( z_0 \in U \) such that

\[\max_{|z| < |z_0|} |w(z)| = |w(z_0)| = 1,\]

and apply Lemma 1, we find that

\[z_0w'(z_0) = kw(z_0) \quad (k \geq 1) \quad (2.5)\]

writing \( w(z) = e^{i\theta} \) \((0 \leq \theta < 2\pi)\) and setting \( z = z_0 \) in (2.4), we get

\[
\left| p + 1 + \frac{z_0 \left( D_\lambda^{\lambda,p} f(z_0) \right)''}{\left( D_\lambda^{\lambda,p} f(z_0) \right)'} - (\mu - 1) \left( p + \frac{z_0 \left( D_\lambda^{\lambda,p} f(z_0) \right)'}{D_\lambda^{\lambda,p} f(z_0)} \right) - \gamma \left( \frac{z_0^{p+1} \left( D_\lambda^{\lambda,p} f(z_0) \right)'}{\left( z_0^p D_\lambda^{\lambda,p} f(z_0) \right)^{\mu-1}} + p \right) \right|
\]
\[ = \left| \gamma(p-\alpha)e^{i\theta} + \frac{(p-\alpha)ke^{i\theta}}{p+(p-\alpha)e^{i\theta}} \right| \]
\[ \geq \Re \left( \gamma(p-\alpha) + \frac{(p-\alpha)k}{p+(p-\alpha)e^{i\theta}} \right) \]
\[ > \gamma(p-\alpha) + \frac{(p-\alpha)}{2p-\alpha} \]
\[ = \frac{(p-\alpha)(\gamma(2p-\alpha) + 1)}{2p-\alpha}, \]

which contradicts our assumption (2.1). Therefore, we have \(|w(z)| < 1\) in \(U\).

Finally, we have
\[ \left| \frac{z^{p+1}(D^{\lambda,p}_{\ast} f(z))'}{(z^{p}D^{\lambda,p}_{\ast} f(z))^{\mu-1}} + p \right| = |(p-\alpha)w(z)| = (p-\alpha)|w(z)| \]
\[ < p-\alpha \quad (z \in U), \quad (2.6) \]

that is \(f(z) \in J_{p}(\lambda, \mu, \alpha)\). This proves the Theorem 1.

**Theorem 2.** Let \(p \in \mathbb{N}, \lambda > -p, \mu \geq 0\) and \(0 \leq \delta < p\). If \(f(z) \in \Sigma(p)\) satisfies the following inequality
\[ \Re \left[ \frac{z^{p+1}(D^{\lambda,p}_{\ast} f(z))'}{(z^{p}D^{\lambda,p}_{\ast} f(z))^{\mu-1}} + (\mu - 1) z \frac{(D^{\lambda,p}_{\ast} f(z))'}{D^{\lambda,p}_{\ast} f(z)} - \frac{z (D^{\lambda,p}_{\ast} f(z))''}{(D^{\lambda,p}_{\ast} f(z))'} - 1 \right] \]
\[ > \delta \left( \frac{1}{2} + \frac{1}{2} \right) + \left( \delta(\mu - 2) - \frac{1}{2} \right) \quad (2.7) \]

then \(f(z) \in J_{p}(\lambda, \mu, \delta)\).

**Proof.** Define the functions \(q(z)\) by
\[ \frac{z^{p+1}(D^{\lambda,p}_{\ast} f(z))'}{(z^{p}D^{\lambda,p}_{\ast} f(z))^{\mu-1}} = -\delta + (\delta - p)q(z), \quad (2.8) \]

then we see that \(q(z) = 1 + q_{1}z + q_{2}z^{2} + ...\) is analytic in \(U\). Now differentiating both sides of (2.8) with respect to \(z\) logarithmically, we get
\[ \delta + (p - \delta)q(z) \left( 1 + \frac{z (D^{\lambda,p}_{\ast} f(z))''}{(D^{\lambda,p}_{\ast} f(z))'} + (1 - \mu) \frac{z (D^{\lambda,p}_{\ast} f(z))'}{D^{\lambda,p}_{\ast} f(z)} \right) \]
Again using (2.8) in (2.10), we find that
\[(p - \delta)zq'(z) + p(\mu - 2)[\delta + (p - \delta)q(z)]. \quad (2.9)\]

From (2.8) and (2.9) we have
\[-\frac{z^{p+1} (D^p_\lambda f(z))'}{(z^p D^p_\lambda f(z))^{\mu-1}} \left(1 + \frac{z (D^p_\lambda f(z))''}{(D^p_\lambda f(z))'} + (1 - \mu) \frac{z (D^p_\lambda f(z))'}{D^p_\lambda f(z)} \right)\]
\[= (p - \delta)zq'(z) + p(\mu - 2)[\delta + (p - \delta)q(z)]. \quad (2.10)\]

Again using (2.8) in (2.10), we find that
\[-\frac{z^{p+1} (D^p_\lambda f(z))'}{(z^p D^p_\lambda f(z))^{\mu-1}} \left(1 + \frac{z (D^p_\lambda f(z))''}{(D^p_\lambda f(z))'} + (1 - \mu) \frac{z (D^p_\lambda f(z))'}{D^p_\lambda f(z)} \right)\]
\[= (p - \delta)zq'(z) + (p - \delta)^2q^2(z) + (p - \delta)[2\delta + p(\mu - 2)]q(z) + p\delta(\mu - 2) + \delta^2\]
\[= \phi(q(z), zq'(z); z),\]

where
\[\phi(r, s, z) = (p - \delta)s + (p - \delta)^2r^2 + (p - \delta)[2\delta + p(\mu - 2)]r + p\delta(\mu - 2) + \delta^2. \quad (2.11)\]

For all real \(x, y\) satisfying \(y \leq -(1 + x^2)/2\), we have
\[\Re (\phi(ix, y, z)) = (p - \delta)y + (p - \delta)^2x^2 + p\delta(\mu - 2) + \delta^2\]
\[\leq -\frac{1}{2}(p - \delta)(1 + x^2) - (p - \delta)^2x^2 + p\delta(\mu - 2) + \delta^2\]
\[= -\frac{1}{2}(p - \delta) - (p - \delta) \left(\frac{1}{2} + p - \delta\right) x^2 + p\delta(\mu - 2) + \delta^2\]
\[\leq \delta p(\mu - 2) + \delta^2 - \frac{1}{2}(p - \delta)\]
\[= \delta \left(\delta + \frac{1}{2}\right) + p \left(\delta(\mu - 2) - \frac{1}{2}\right).\]

Let
\[S = \left\{ w : \Re(w) > \delta \left(\delta + \frac{1}{2}\right) + p \left(\delta(\mu - 2) - \frac{1}{2}\right) \right\},\]

then \(\phi(q(z), zq'(z); z) \in S\) and \(\phi(ix, y, z) \notin S\) for all real \(x\) and \(y < -(1 + x^2)/2\), \(z \in \mathbb{U}\). By using Lemma 2, we have \(\Re(q(z)) > 0\), that is \(f(z) \in \mathcal{J}_p(\lambda, \mu, \delta)\). This proves the Theorem 2.
3. Some Consequences Of Main Results

Among various interesting and important consequences of our Theorems 1 and 2 we mentioned here some of the corollaries relating subclasses $\Sigma^*(\alpha), \Sigma_c(\alpha), \Sigma^*$ and $\Sigma_c$ which are easily deducible from the main results.

Firstly, if we take $\lambda = 0$ and $\mu = \gamma = p = 1$, then Theorem 1 gives the following result

**Corollary 1.** If $f(z) \in \Sigma$ satisfies the following inequality

$$\left| \frac{z f''(z)}{f'(z)} - z^2 f'(z) + 1 \right| < \frac{(1 - \alpha)(3 - \alpha)}{2 - \alpha} \quad (0 \leq \alpha < 1), \quad (3.1)$$

then $f(z) \in \Sigma_c(\alpha)$.

Further setting $\alpha = 0$ in Corollary 1, we get

**Corollary 2.** If $f(z) \in \Sigma$ satisfies the following inequality

$$\left| \frac{z f''(z)}{f'(z)} - z^2 f'(z) + 1 \right| < \frac{3}{2}, \quad (3.2)$$

then $f(z) \in \Sigma_c (\Sigma_c = \Sigma_c(0))$.

Again if we set $\lambda = \gamma = \alpha = 0$ and $\mu = p = 1$ in theorem 1, we get

**Corollary 3.** If $f(z) \in \Sigma$ satisfies the following inequality

$$\left| \frac{z f''(z)}{f'(z)} + 2 \right| < \frac{1}{2}, \quad (3.3)$$

then $f(z) \in \Sigma_c$.

Also for $\lambda = \delta = 0$ and $\mu = p = 1$, Theorem 2 gives

**Corollary 4.** If $f(z) \in \Sigma$ satisfies the following inequality

$$\Re \left[ z^2 \{ f'(z)(z^2 f'(z) - 1) - z f''(z) \} \right] > -\frac{1}{2}, \quad (3.4)$$

then $f(z) \in \Sigma_c$.

Again on setting $\mu = 2, \gamma = p = 1$ and $\lambda = 0$ in Theorem 1, we get
Corollary 5. If \( f(z) \in \Sigma \) satisfies the following inequality
\[
\left| \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)} \right| < \frac{(1-\alpha)(3-\alpha)}{2-\alpha} \quad (0 \leq \alpha < 1), \tag{3.5}
\]
then \( f(z) \in \Sigma^*(\alpha) \).

On further setting \( \alpha = 0 \) in Corollary 5, we get

Corollary 6. If \( f(z) \in \Sigma \) satisfies the following inequality
\[
\left| \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)} \right| < \frac{3}{2}, \tag{3.6}
\]
then \( f(z) \in \Sigma^* (\Sigma^* = \Sigma^*(0)) \).

Also let \( \mu = 2, p = 1 \) and \( \gamma = \alpha = \lambda = 0 \) in Theorem 1, we have

Corollary 7. If \( f(z) \in \Sigma \) satisfies the following inequality
\[
\left| \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1 \right| < \frac{1}{2}, \tag{3.7}
\]
then \( f(z) \in \Sigma^* \).

For \( \mu = 2, p = 1 \) and \( \delta = \lambda = 0 \), Theorem 2 gives

Corollary 8. If \( f(z) \in \Sigma \) satisfies the following inequality
\[
\Re \left[ \frac{zf'(z)}{f(z)} \left\{ \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} - 1 \right\} \right] > -\frac{1}{2}, \tag{3.8}
\]
then \( f(z) \in \Sigma^* \).

References


