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A Companion for the Generalized Ostrowski and the Generalized Trapezoid Type Inequalities *

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Abstract

In this paper we will establish a companion for the generalized Ostrowski and the generalized trapezoid inequalities, including functions of bounded variation, Lipschitzian, convex and absolutely continuous functions, which generalizes Barnett et al.'s some results (N.S. Barnett et al., Math. Comput. Modeling 50 (2009) 179-187). Applications for weighted means are also given.

Keywords and Phrases: *Generalized Ostrowski type inequalities, Generalized trapezoid type inequalities, Riemann-Stieltjes integral.*

1. Introduction

In 2000, Dragomir [2] answered to the problem of approximating the Stieltjes integral $\int_a^b f(x)du(x)$ by the quantity $[u(b) - u(a)]f(x)$, which is a natural generalization of the Ostrowski problem [3] analysed in 1937. He obtained the

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following result:

$$\begin{aligned}
&\left|\int_a^b f(t)du(x) - [u(b)-u(a)]f(x)\right| \\
&\quad \le H\left[(x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f)\right] \qquad (1.1)\\
&\quad \le H \begin{cases} [(x-a)^r + (b-x)^r]\left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right]; \\ [(x-a)^{qr} + (b-x)^{qr}]^{\frac{1}{q}}\left[\left(\bigvee_a^x(f)\right)^p + \left(\bigvee_x^b(f)\right)^p\right]^{\frac{1}{p}}, \\ \text{if } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b(f), \end{cases}
\end{aligned}$$

for each $x \in [a,b]$, provided f is of bounded variation on $[a,b]$, while $u : [a,b] \to \mathbb{R}$ is r-H-Hölder continuous, i.e., we recall that:

$$|u(x)-u(y)| \le H|x-y|^r \quad \text{for each } x,y \in [a,b].$$

From a different view point, the problem of approximating the Stieltjes integral $\int_a^b f(x)du(x)$ by the generalized trapezoid rule $[(u(b)-u(x))f(b) + (u(x)-u(a))f(a)]$ was considered by Dragomir et al. [4]. The following inequality was obtained:

$$\begin{aligned}
&\left|\int_a^b f(x)du(x) - [(u(b)-u(x))f(b) + (u(x)-u(a))f(a)]\right| \\
&\qquad \le H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b(f) \le H(b-a)^r \bigvee_a^b(f)
\end{aligned}$$

for each $x \in [a,b]$, provided f is of bounded variation on $[a,b]$ while $u : [a,b] \to \mathbb{R}$ is r-H-Hölder continuous.

For a Riemann-Stieltjes integrable function $f : [a,b] \to \mathbb{R}$ and for a given $x \in [a,b]$, it is natural to investigate the distances between the quantities

$$f(x), \ \frac{1}{u(b)-u(a)}\int_a^b f(x)du(x) \text{ and } \frac{(u(b)-u(x))f(b) + (u(x)-u(a))f(a)}{u(b)-u(a)} \qquad (1.2)$$

respectively, and to seek sharp upper bounds for these distances in terms of different measure that can be associated with f, where f is restricted to particular classes of functions including functions of bounded variation, Lipschitzian, convex and absolutely continuous functions.

The authors of [2, 4] have been given sharp upper bounds for absolute value between the first quantity and the second, the second and the third in (1.2).

The main aim of this paper is to provide sharp upper bounds for absolute value of the remaining difference between the first quantity and the third in (1.2), that is,

$$\Psi_f(x) := f(x) - \frac{(u(b)-u(x))f(b)+(u(x)-u(a))f(a)}{u(b)-u(a)}, \quad x \in [a,b]. \tag{1.3}$$

As applications, some bounds for the absolute value of the difference

$$\Phi_f(x) := \sum_{i=1}^{n} p_i f(x_i) - \frac{(u(b)-\sum_{i=1}^{n} p_i u(x_i))f(b)+(\sum_{i=1}^{n} p_i u(x_i)-u(a))f(a)}{u(b)-u(a)}, \tag{1.4}$$

where $x_i \in [a,b]$, $p_i \geq 0, i \in \{1,2,\cdots,n\}$ and $\Sigma_{i=1}^{n} p_i = 1$, are also given.

Remark Using the Stieltjes integral by Dragomir [2], generalization of the Ostrowski problem [3] was considered, so our results are natural to generalize some results obtained by Barnett et al.'s some results [1].

2. The case when f is of bounded variation and u Hölder continuous

The following representation holds.

Lemma 2.1 *Let f is of bounded function on $[a,b]$ and let $T : [a,b]^2 \to \mathbb{R}$ be given by*

$$T(x,s) := \begin{cases} u(x)-u(a), & if \;\; s \in [a,x], \\ u(x)-u(b), & if \;\; s \in [x,b]. \end{cases} \tag{2.1}$$

Then we have the following representation,

$$\Psi_f(x) = \frac{1}{u(b) - u(a)} \int_a^b T(x,s) df(s), \quad x \in [a,b], \tag{2.2}$$

where the integral is considered in the Riemann-Stieltjes sense.

Proof. If f is bounded on $[a,b]$, the for any $t \in [a,b]$, the Riemann-Stieltjes integral $\int_a^x df(s) = f(x) - f(a)$, $\int_x^b df(s) = f(b) - f(x)$. It follows that

$$\begin{aligned} \int_a^b T(x,s)df(s) &= (u(x)-u(a)) \int_a^x df(s) + (u(x) - u(b)) \int_x^b df(s) \\ &= (u(b)-u(a))\Psi_f(x), \end{aligned}$$

for any $t \in [a,b]$. $\square$

The following provides a sharp bound for the absolute value of Ψ_f where f is of bounded variation and u is r-H-Hölder continuous.

Theorem 2.2 *If $f : [a,b] \to \mathbb{R}$ is of bounded variation and $u : [a,b] \to \mathbb{R}$ is r-H-Hölder continuous on the interval $[a,b]$, i.e.,*

$$|u(x) - u(y)| \leq H|x-y|^r \quad \text{for each } x,y \in [a,b].$$

Then

$$\begin{aligned} |\Psi_f(x)| &\leq \frac{1}{|u(b)-u(a)|} \left[|u(x)-u(a)| \bigvee_a^x (f) + |u(x)-u(b)| \bigvee_x^b (f) \right] && (2.3) \\ &\leq \frac{H}{|u(b)-u(a)|} \left[(x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f) \right] && (2.4) \\ &\leq \frac{H}{|u(b)-u(a)|} \begin{cases} [(x-a)^r + (b-x)^r] \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right]; \\ [(x-a)^{qr} + (b-x)^{qr}]^{\frac{1}{q}} \left[\left(\bigvee_a^x (f) \right)^p + \left(\bigvee_x^b (f) \right)^p \right]^{\frac{1}{p}}, \\ \text{if } \ p>1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^r \bigvee_a^b (f), \end{cases} && (2.5) \end{aligned}$$

for any $x \in [a,b]$*. The constant* $1/2$ *is also the best possible in both branches of (2.5).*

Proof. Utilizing the representation (2.2), we have

$$\begin{aligned} |\Psi_f(x)| &= \frac{1}{|u(b)-u(a)|}\left|(u(x)-u(a))\int_a^x df(s) + (u(x)-u(b))\int_x^b df(s)\right| \\ &\leq \frac{1}{|u(b)-u(a)|}\left[|u(x)-u(a)|\left|\int_a^x df(s)\right| + |u(x)-u(b)|\left|\int_x^b df(s)\right|\right] \\ &\leq \frac{1}{|u(b)-u(a)|}\left[|u(x)-u(a)|\bigvee_a^x(f) + |u(x)-u(b)|\bigvee_x^b(f)\right] \\ &\leq \frac{H}{|u(b)-u(a)|}\left[(x-a)^r\bigvee_a^x(f) + (b-x)^r\bigvee_x^b(f)\right] \end{aligned}$$

which implies the inequalities (2.3) and (2.4).

Combination inequality (1.1) and the above inequality, we have inequality (2.5).

Now, we prove that The constant $1/2$ is also the best possible in both branches of (2.5). Consider the function $f_0(t) = |t-(a+b)/2|$ which is of bounded variation on $[a,b]$, with $f_0(a) = f_0(b) = (b-a)/2$ and $\bigvee_a^b(f_0) = b-a$. And $u_0(x) = x$ which is 1-1-Hölder continuous. According to the proof of the best possibility of the constant in Theorem 1 in [1], the sharpness of the constant $1/2$ in the inequality (2.5) is the best possible. □

As application, we give the case when f and u have some slight variations as follows.

Corollary 2.3 *If* $f : [a,b] \to \mathbb{R}$ *is* L_1*-Lipschitzian on* $[a,x]$ *and* L_2*-Lipschitzian on* $[x,b]$*,* $L_1, L_2 > 0$*,* $x \in [a,b]$*, while the function* $u : [a,b] \to \mathbb{R}$ *satisfies some local Hölder continuous, namely,*

$$|u(t)-u(a)| \leq H_1|t-a|^{r_1} \quad \text{for any } t \in [a,x] \tag{2.6}$$

and

$$|u(b)-u(t)| \leq H_2|b-t|^{r_2} \quad \text{for any } t \in [x,b] \tag{2.7}$$

where $H_1, H_2 > 0$, $r_1, r_2 \in (-1, +\infty)$, *then*

$$
\begin{aligned}
|\Psi_f(x)| &\leq \frac{1}{|u(b)-u(a)|}\left[L_1|u(x)-u(a)|(x-a)+L_2|u(x)-u(b)|(b-x)\right] &(2.8)\\
&\leq \frac{1}{|u(b)-u(a)|}\left[H_1L_1(x-a)^{r_1+1}+H_2L_2(b-x)^{r_2+1}\right] &(2.9)\\
&\leq \frac{1}{|u(b)-u(a)|}\begin{cases}\max\{H_1L_1, H_2L_2\}\left[(x-a)^{r_1+1}+(b-x)^{r_2+1}\right];\\ \left[(H_1L_1)^p+(H_2L_2)^p\right]^{\frac{1}{p}}\left[(x-a)^{qr_1}+(b-x)^{qr_2}\right]^{\frac{1}{q}},\\ \text{if } \ p>1, \frac{1}{p}+\frac{1}{q}=1;\\ \max\{(x-a)^{r_1+1},(b-x)^{r_2+1}\}(H_1L_1+H_2L_2),\end{cases} &(2.10)
\end{aligned}
$$

for any $x \in [a, b]$.

Proof. It is well known that if $g : [\alpha, \beta] \to \mathbb{R}$ is L-Lipschitzian, then g is of bounded variation and $\bigvee_{\alpha}^{\beta}(g) \leq L(\beta-\alpha)$. Therefore, by the first inequality (2.4), we get the corresponding inequality (2.8). Using the local Hölder continuity of the function u, we have inequality (2.9) from (2.8). The other inequalities follow by the Hölder inequality and the details are omitted. □

Corollary 2.4 *If* $f : [a, b] \to \mathbb{R}$ *is monotonic nondecreasing, while* $u : [a, b] \to \mathbb{R}$ *is* L*-Lipschitzian on* $[a, b]$*, where* $L > 0$*, then*

$$
\begin{aligned}
|\Psi_f(x)| &\leq \frac{1}{|u(b)-u(a)|}\left[|u(x)-u(a)|(f(x)-f(a))+|u(x)-u(b)|(f(b)-f(x))\right]\\
&\leq \frac{L}{|u(b)-u(a)|}\left[(x-a)(f(x)-f(a))+(b-x)(f(b)-f(x))\right]\\
&\leq \frac{L}{|u(b)-u(a)|}\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[f(b)-f(a)\right];\\ \left[(x-a)^p+(b-x)^p\right]^{\frac{1}{p}}\left[(f(x)-f(a))^q+(f(b)-f(x))^q\right]^{\frac{1}{q}},\\ \text{if } \ p>1, \frac{1}{p}+\frac{1}{q}=1;\\ (b-a)\left[\frac{1}{2}(f(b)-f(a))+\left|f(x)-\frac{f(a)+f(b)}{2}\right|\right],\end{cases}
\end{aligned}
$$

for any $x \in [a, b]$.

Proof. It is easy to observe that we obtain Corollary 2.4 by using Theorem 2.2 and Hölder inequality, so the details are omitted. □

3. The case when f is absolutely continuous and u Hölder continuous

When f is absolutely continuous, the following representation holds.

Lemma 3.1 *If f is of bounded function on $[a,b]$. Then we have the following representation,*

$$\Psi_f(x) = \frac{1}{u(b)-u(a)} \int_a^b T(x,s) f'(s) ds, \quad x \in [a,b], \tag{3.1}$$

where the integral is considered in the Lebesgue sense and where the kernel $T : [a,b]^2 \to \mathbb{R}$ has been defined in (2.1).

We cite the following Lebesgue norms defined in Section 3 in [1] as follows.

$$\| f' \|_{[a,b],\infty} := ess \sup_{x\in[a,b]} |f'(x)|, \qquad \| f' \|_{[a,b],p} := \left(\int_a^b |f'(x)|^p dx \right)^{\frac{1}{p}}, \quad p \geq 1.$$

Theorem 3.2 *If f is absolutely continuous on $[a,b]$, $u : [a,b] \to \mathbb{R}$ is r-H-Hölder continuous on $[a,b]$, where $H > 0$ and $r \in (-1,\infty)$. Then we have the following inequalities:*

$$\begin{aligned}
|\Psi_f(x)| &\leq \frac{1}{|u(b)-u(a)|} \left[|u(x)-u(a)| \, \| f' \|_{[a,x],1} + |u(b)-u(x)| \, \| f' \|_{[x,b],1} \right] && (3.2)\\
&\leq \frac{H}{|u(b)-u(a)|} \left[(x-a)^r \, \| f' \|_{[a,x],1} + (b-x)^r \, \| f' \|_{[x,b],1} \right] && (3.3)\\
&\leq \frac{H}{|u(b)-u(a)|} W(x), \quad x \in [a,b], && (3.4)
\end{aligned}$$

where $W(x)$ is defined by

$$\begin{aligned}
W(x) \; &:= \; \begin{cases} (x-a)^{r+1} \, \| f' \|_{[a,x],\infty}, & if \;\; f' \in L_\infty[a,b] \\ (x-a)^{r+\frac{1}{q}} \, \| f' \|_{[a,x],p}, & if \;\; f' \in L_p[a,b], \;\; p > 1, \frac{1}{p} + \frac{1}{q} = 1; \end{cases} \\
&+ \begin{cases} (b-x)^{r+1} \, \| f' \|_{[x,b],\infty}, & if \;\; f' \in L_\infty[a,b], \\ (b-x)^{r+\frac{1}{\beta}} \, \| f' \|_{[x,b],\alpha}, & if \;\; f' \in L_\alpha[a,b], \; \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \end{cases}
\end{aligned}$$

and $W(x)$ should be seen as all four possible combinations.

Proof. By Lemma 3.1, we have

$$\begin{aligned}
|\Psi_f(x)| &= \left|\frac{1}{u(b)-u(a)}\left[(u(x)-u(a))(f(x)-f(a))+(u(b)-u(x))(f(b)-f(x))\right]\right| \\
&= \left|\frac{1}{u(b)-u(a)}\left[(u(x)-u(a))\int_a^x f'(s)ds+(u(b)-u(x))\int_x^b f'(s)ds\right]\right| \\
&\leq \frac{1}{|u(b)-u(a)|}\left[|u(x)-u(a)|\int_a^x |f'(s)|ds+|u(b)-u(x)|\int_x^b |f'(s)|ds\right] \\
&\leq \frac{1}{|u(b)-u(a)|}\left[|u(x)-u(a)| \parallel f' \parallel_{[a,x],1} +|u(b)-u(x)| \parallel f' \parallel_{[x,b],1}\right] \\
&\leq \frac{H}{|u(b)-u(a)|}\left[(x-a)^r \parallel f' \parallel_{[a,x],1} +(b-x)^r \parallel f' \parallel_{[x,b],1}\right],
\end{aligned}$$

for $x \in [a,b]$, which implies inequalities (3.2) and (3.3).

Utilizing (3.4) and (3.5) in [1] and the above inequality, we obtain the desired inequality (3.4). □

Corollary 3.3 *If f is absolutely continuous on $[a,b]$, the function $u : [a,b] \to \mathbb{R}$ satisfies some local Hölder continuous defined by (2.6) and (2.7). Then we have*

$$\begin{aligned}
|\Psi_f(x)| &\leq \frac{1}{|u(b)-u(a)|}\left[|u(x)-u(a)| \parallel f' \parallel_{[a,x],1} +|u(b)-u(x)| \parallel f' \parallel_{[x,b],1}\right] \\
&\leq \frac{1}{|u(b)-u(a)|}\left[H_1(x-a)^{r_1} \parallel f' \parallel_{[a,x],1} +H_2(b-x)^{r_2} \parallel f' \parallel_{[x,b],1}\right] \\
&\leq \frac{1}{|u(b)-u(a)|}W(x), \quad x \in [a,b],
\end{aligned}$$

where $W(x)$ is defined by

$$\begin{aligned}
W(x) &:= \begin{cases} H_1(x-a)^{r_1+1} \parallel f' \parallel_{[a,x],\infty}, & if \ \ f' \in L_\infty[a,b] \\ H_1(x-a)^{r_1+\frac{1}{q}} \parallel f' \parallel_{[a,x],p}, & if \ \ f' \in L_p[a,b], \ \ p>1, \frac{1}{p}+\frac{1}{q}=1; \end{cases} \\
&\quad + \begin{cases} H_2(b-x)^{r_2+1} \parallel f' \parallel_{[x,b],\infty}, & if \ \ f' \in L_\infty[a,b], \\ H_2(b-x)^{r_2+\frac{1}{\beta}} \parallel f' \parallel_{[x,b],\alpha}, & if \ \ f' \in L_\alpha[a,b], \ \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1; \end{cases}
\end{aligned}$$

and $W(x)$ should be seen as all four possible combinations.

Proof. It is similar to the proof of Theorem 3.2, so the details are omitted. □

4. The case when f is convex and u monotonic nondecreasing and bi-Hölder

Before giving the case when f is convex and u is monotonic nondecreasing and bi-Hölder, we establish sharp lower and upper bounds for the remaining differences as follows:

$$\Omega_1(x) := \int_a^b f(x)du(x) - (u(b) - u(a))f(x) \tag{4.1}$$

and

$$\Omega_2(x) := [(u(b) - u(x))f(b) + (u(x) - u(a))f(a)] - \int_a^b f(x)du(x). \tag{4.2}$$

Theorem 4.1 *If $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$ with $f'_-(b)$ and $f'_+(a)$ finite, and $u : [a, b] \to \mathbb{R}$ is monotonic nondecreasing and bi-Hölder function on $[a, b]$, that is,*

$$\mathscr{L}_1(y - x)^r \le u(y) - u(x) \le \mathscr{L}_2(y - x)^r, \quad \text{for} \quad x \le y, \quad x, y \in [a, b], \tag{4.3}$$

where $\mathscr{L}_1, \mathscr{L}_2 > 0$ and $r > -1$. Then we have the following inequalities:

$$\begin{aligned}\frac{1}{r+1}&[\mathscr{L}_1(b - x)^{r+1} f'_+(x) - \mathscr{L}_2(x - a)^{r+1} f'_-(x)] \\ &\le \Omega_1(x) \le \frac{1}{r+1}[\mathscr{L}_2(b - x)^{r+1} f'_-(b) - \mathscr{L}_1(x - a)^{r+1} f'_+(a)]\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\frac{1}{r+1}&[\mathscr{L}_1(b - x)^{r+1} f'_+(x) - \mathscr{L}_2(x - a)^{r+1} f'_-(x)] \\ &\le \Omega_2(x) \le \frac{1}{r+1}[\mathscr{L}_2(b - x)^{r+1} f'_-(b) - \mathscr{L}_1(x - a)^{r+1} f'_+(a)],\end{aligned} \tag{4.5}$$

where $\Omega_1(x)$ and $\Omega_2(x)$ are defined by (4.1) and (4.2). The constant $1/(r + 1)$ is sharp in both inequalities.

Proof. First of all, we give the proof of inequality (4.4). It is easy to see that for any locally absolutely continuous function $f : [a, b] \to \mathbb{R}$, we have the identity

$$\int_a^x (u(t) - u(a))f'(t)dt + \int_x^b (u(t) - u(b))f'(t)dt$$
$$= (u(b) - u(a))f(x) - \int_a^b f(t)du(t) \tag{4.6}$$

for any $x \in (a, b)$, where f' is the derivation of f which exists a.e. on (a, b).

Since f is convex, then it is locally Lipschitzian and thus (4.6) holds. Moreover, for any $x \in (a, b)$, we have the inequalities

$$f'(t) \leq f'_-(x) \quad \text{for a.e.} \quad t \in [a, x] \tag{4.7}$$

and

$$f'(t) \geq f'_+(x) \quad \text{for a.e.} \quad t \in [x, b]. \tag{4.8}$$

If we multiply (4.7) by $u(t) - u(a) \geq 0$, $t \in [a, x]$ and integrate on $[a, x]$, by (4.3), we get

$$\int_a^x (u(t) - u(a))f'(t)dt \leq f'_-(x) \int_a^x (u(t) - u(a))dt \leq \frac{1}{r+1}\mathscr{L}_2(x-a)^{r+1} f'_-(x) \tag{4.9}$$

and if we multiply (4.8) by $u(b) - u(x) \geq 0$, $t \in [x, b]$ and integrate on $[x, b]$, by (4.3), we get

$$\int_x^b (u(b) - u(t))f'(t)dt \geq f'_+(x) \int_x^b (u(b) - u(t))dt \geq \frac{1}{r+1}\mathscr{L}_1(b-x)^{r+1} f'_+(x). \tag{4.10}$$

If we subtract (4.10) from (4.9) and use the representation (4.6), we deduce the first inequality in (4.4).

Since f is convex, then we have the inequalities

$$f'(t) \geq f'_+(a) \quad \text{for a.e.} \quad t \in [a, x] \tag{4.11}$$

and

$$f'(t) \leq f'_-(b) \quad \text{for a.e.} \quad t \in [x, b]. \tag{4.12}$$

If we multiply (4.11) by $u(t)-u(a)\geq 0$, $t\in[a,x]$ and integrate on $[a,x]$, by (4.3), we get

$$\int_a^x (u(t)-u(a))f'(t)dt \geq f'_+(a)\int_a^x (u(t)-u(a))dt \geq \frac{1}{r+1}\mathscr{L}_1(x-a)^{r+1}f'_+(a) \quad (4.13)$$

and if we multiply (4.12) by $u(b)-u(x)\geq 0$, $t\in[x,b]$, integrate on $[x,b]$ and integrate on $[a,x]$, by (4.3), we get

$$\int_x^b (u(b)-u(t))f'(t)dt \leq f'_-(b)\int_x^b (u(b)-u(t))dt \leq \frac{1}{r+1}\mathscr{L}_2(b-x)^{r+1}f'_-(b). \quad (4.14)$$

If we subtract (4.14) from (4.13) and use the representation (4.6), we deduce the second inequality in (4.4).

Now we prove that the constant $1/(r+1)$ is also the best possible in inequalities (4.4). Consider the function $f_0(t)=k\,|t-(a+b)/2|$ which is a convex function on the interval $[a,b]$, where $k>0$, $t\in[a,b]$. Then

$$f'_{0-}\left(\frac{a+b}{2}\right)=-k,\quad f'_{0+}\left(\frac{a+b}{2}\right)=k \quad \text{and} \quad f_0\left(\frac{a+b}{2}\right)=0.$$

And $u_0(x)=x$, then $\mathscr{L}_1=\mathscr{L}_2=r=1$. Thus we have $\int_a^b f_0(t)dt=k(b-a)^2/2$. If in (4.4) we choose $f=f_0$, $u=u_0$ and $x=(a+b)/2$. According to the proof of the best possibility of the constant in Lemma 2.1 in [5], the sharpness of the constant $1/(r+1)$ in the inequality (4.4) is the best possible.

Secondly, we give the proof of inequality (4.5). It is easy to see that for any locally absolutely continuous function $f:[a,b]\to\mathbb{R}$, we have the identity

$$\begin{aligned}\int_a^b (u(t)-u(x))f'(t)dt &= (u(b)-u(x))f(b)\\ &\quad +(u(x)-u(a))f(a)-\int_a^b f(t)du(t) \quad (4.15)\end{aligned}$$

for any $x\in(a,b)$, where f' is the derivation of f which exists a.e. on (a,b).

Since f is convex, then it is locally Lipschitzian and thus (4.15) holds. The following proof is similar to the proof of inequalities (4.4) and Lemma 2.1 in [6], so the details are omitted. □

In the following we give sharp lower and upper bounds for the remaining difference (1.4) when f is convex and u is monotonic nondecreasing and bi-Hölder.

Theorem 4.2 *If $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$ with $f'_-(b)$ and $f'_+(a)$ finite, and $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing and bi-Hölder function on $[a,b]$ defined by (4.3). Then we have the following inequalities:*

$$\begin{aligned}\frac{1}{u(b)-u(a)}&[\mathscr{L}_1(x-a)^{r+1}f'_+(a)-\mathscr{L}_2(b-x)^{r+1}f'_-(b)] \le \Psi_f(x)\\ &\le \frac{1}{u(b)-u(a)}[\mathscr{L}_2(x-a)^{r+1}f'_-(x)-\mathscr{L}_1(b-x)^{r+1}f'_+(x)] \qquad (4.16)\end{aligned}$$

where $\Psi_f(x)$ is defined by (1.3). The constant 1 is the best possible on both sides of (4.16).

Proof. From Lemma 2.1,

$$\begin{aligned}(u(b)-u(a))\Phi_f(x) &= (u(x)-u(a))(f(x)-f(a))\\ &\quad -(u(b)-u(x))(f(b)-f(x)), \quad x \in [a,b]. \qquad (4.17)\end{aligned}$$

Let $x \in (a,b)$, then, by the convexity of f, we have

$$(x-a)f'_-(x) \ge f(x)-f(a) \ge (x-a)f'_+(a) \qquad (4.18)$$

and

$$(b-x)f'_-(b) \ge f(b)-f(x) \ge (b-x)f'_+(x). \qquad (4.19)$$

If we multiply (4.18) by $u(x)-u(a) > 0$ and (4.19) by $u(b)-u(x) > 0$, we obtain

$$\begin{aligned}(u(x)-u(a))(x-a)f'_-(x) &\ge (u(x)-u(a))(f(x)-f(a))\\ &\ge (u(x)-u(a))(x-a)f'_+(a) \qquad (4.20)\end{aligned}$$

and

$$\begin{aligned}(u(b)-u(x))(b-x)f'_-(b) &\ge (u(b)-u(x))(f(b)-f(x))\\ &\ge (u(b)-u(x))(b-x)f'_+(x). \qquad (4.21)\end{aligned}$$

By (4.3), the above inequalities can rewrite

$$\mathscr{L}_2(x-a)^{r+1}f'_-(x) \ge (u(x)-u(a))(f(x)-f(a)) \ge \mathscr{L}_1(x-a)^{r+1}f'_+(a) \quad (4.22)$$

and

$$-\mathscr{L}_1(b-x)^{r+1}f'_+(x) \geq -(u(b)-u(x))(f(b)-f(x)) \geq -\mathscr{L}_2(b-x)^{r+1}f'_-(b). \quad (4.23)$$

Finally, on adding (4.22) to (4.23), we deduce the desired result (4.16).

Now we prove that The constant 1 is also the best possible in inequalities (4.16). Consider the function $f_0(t) = k\,|t-(a+b)/2|$ which is a convex function on $[a,b]$, where $k>0$, $t \in [a,b]$. Then

$$f'_{0^-}(b) = -k, \quad f'_{0^+}(a) = k, \quad f'_{0^-}\left(\frac{a+b}{2}\right) = -k, \quad f'_{0^+}\left(\frac{a+b}{2}\right) = k,$$

$$f_0\left(\frac{a+b}{2}\right) = 0 \quad \text{and} \quad f_0(a) = f_0(b) = \frac{k(b-a)}{2}.$$

And $u_0(x) = x$, then $\mathscr{L}_1 = \mathscr{L}_2 = r = 1$. If in (4.4) we choose $f = f_0$, $u = u_0$ and $x = (a+b)/2$. According to the proof of the best possibility of the constant in Theorem 3 in [1], the sharpness of the constant 1 in the inequality (4.16) is the best possible. □

5. Some applications

As applications, some bounds for the absolute value of the difference (1.4).

Proposition 5.1 *If $f:[a,b]\to\mathbb{R}$ is of bounded variation and $u:[a,b]\to\mathbb{R}$ is r-H-Hölder continuous on $[a,b]$. Then*

$$|\Phi_f(x)| \leq \frac{H}{|u(b)-u(a)|}\left[\frac{1}{2}(b-a)+\sum_{i=1}^{n} p_i\left|x_i-\frac{a+b}{2}\right|\right]^r \bigvee_a^b(f), \qquad (5.1)$$

where $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$, the constant $1/2$ is also the best possible in both branches of (5.1).

Proof. We use the third inequality in (2.5) to state:

$$\left|f(x_i) - \frac{(u(b)-u(x_i))f(b)+(u(x_i)-u(a))f(a)}{u(b)-u(a)}\right| \leq \frac{H}{|u(b)-u(a)|}\left[\frac{1}{2}(b-a)+\left|x_i-\frac{a+b}{2}\right|\right]^r \bigvee_a^b(f) \qquad (5.2)$$

for $i = 1, 2, \cdots, n$.

If we multiply (5.2) by $p_i \geq 0$, sum over $i = 1$ to n, we deduce the desired result (5.1).

The fact that 1/2 is the best possible follows from the fact that it is the best possible for $n = 1$. $\square$

In a similar manner, on utilizing the third inequality in (2.10), we can state the following result:

Proposition 5.2 *If $f : [a,b] \to \mathbb{R}$ is L-Lipschitzian on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ is r-H-Hölder continuous on $[a,b]$. Then*

$$|\Phi_f(x)| \leq \frac{HL(b-a)}{|u(b)-u(a)|}\left[\frac{1}{2}(b-a) + \sum_{i=1}^{n} p_i \left| x_i - \frac{a+b}{2} \right| \right]^r,$$

where $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$.

Finally, on utilizing the inequality in (3.3), we can also state that:

Proposition 5.3 *If f is absolutely continuous on $[a,b]$, $u : [a,b] \to \mathbb{R}$ is r-H-Hölder continuous on $[a,b]$. Then*

$$|\Phi_f(x)| \leq \frac{H}{|u(b)-u(a)|}\left[\sum_{i=1}^{n}(x_i-a)^r \parallel f' \parallel_{[a,x_i],1} + \sum_{i=1}^{n}(b-x_i)^r \parallel f' \parallel_{[x_i,b],1}\right],$$

where $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$.

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On W_2-Curvature Tensor in a Kenmotsu Manifold*

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Abstract

The purpose of this paper is to study some properties of W_2-curvature tensor in Riemannian and Kenmotsu manifolds.

Keywords and phrases: *Riemannian manifold, Kenmotsu manifold, W_2-curvature tensor, Irrotational W_2-curvature tensor, η-Einstein manifold.*

1. Introduction

In 1958, Boothby and Wong [1] studied odd dimensional manifolds with contact and almost contact structures from topological point of view. Sasaki and Hatakeyama [11] re-investigated them using tensor calculus in 1961. In 1972, K. Kenmotsu studied a class of contact Riemannian manifold and call them Kenmotsu manifold [7]. He proved that if Kenmotsu manifold satisfies the condition R(X, Y).R = 0, then the

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manifold is of negative curvature -1, where R is the Riemannian curvature tensor of type (1, 3) and R(X, Y) denotes the derivation of the tensor algebra at each point of the tangent space. The properties of Kenmotsu manifold have been studied by several authors such as De [4], Sinha and Shrivastava [12], Jun, De and Pathak [6], De and Pathak [3], De, Yildiz and Yaliniz [5], Özgur and De [10] and many others. In this paper, we consider Kenmotsu manifold satisfying the conditions $R(\xi, X).W_2 = 0$, $W_2(\xi, X).R = 0$, $P(\xi, X).W_2 = 0$ and $W_2(\xi, X).P = 0$, where W_2 and P denotes the W_2-curvature tensor and projective curvature tensor respectively. Also we have studied the W_2-curvature tensor in a Riemannian manifold and obtained the relation between different curvature tensors. In last section, we have shown that the W_2-curvature tensor in a Kenmotsu manifold M^n is irrotational if and only if $R(X, Y)Z = g(X, Z)Y - g(Y, Z)X + \frac{1}{n-1}\{\eta(X)(\nabla_Z Q)(Y) - \eta(Y)(\nabla_Z Q)(X)\}$.

2. Preliminaries

If on an odd dimensional differentiable manifold M^n ($n = 2m+1$), of differentiability class C^{r+1}, there exists a vector valued real linear function φ, a 1-form η, the associated vector field ξ and the Riemannian metric g satisfying

$$\varphi^2 X = -X + \eta(X)\xi, \tag{1}$$

$$\eta(\varphi X) = 0, \tag{2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\,\eta(Y), \tag{3}$$

for arbitrary vector fields X and Y, then (M^n, g) is said to be an almost contact metric manifold [2] and the structure (φ, ξ, η, g) is called an almost contact metric structure to M^n.

In view of equations (1), (2) and (3), we have

$$\eta(\xi) = 1,\ g(X, \xi) = \eta(X),\ \varphi(\xi) = 0. \tag{4}$$

An almost contact metric manifold is called Kenmotsu manifold [7] if

$$(\nabla_X \varphi) = -\eta(Y)\varphi X - g(X, \varphi Y)\xi, \tag{5}$$

$$(\nabla_X \xi) = X - \eta(X)\xi, \tag{6}$$

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\,\eta(Y), \tag{7}$$

where ∇ is the Levi-Civita connection of g. Also the following relations hold in Kenmotsu manifold [3], [5], [6]

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{8}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi = - R(X, \xi)Y, \tag{9}$$

$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \tag{10}$$

$$S(X, \xi) = -(n-1)\,\eta(X), \tag{11}$$

$$Q\xi = -(n-1)\xi, \tag{12}$$

where Q is the Ricci operator, i.e. $g(QX, Y) = S(X, Y)$ and

$$S(\varphi X, \varphi Y) = S(X, Y) + (n-1)\,\eta(X)\eta(Y), \tag{13}$$

for arbitrary vector fields X, Y, Z on M^n.

A Kenmotsu manifold is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\,\eta(X)\eta(Y) \tag{14}$$

for arbitrary vector fields X and Y, where a and b are smooth functions on M^n.

Projective curvature tensor P, concircular curvature tensor C and the conformal curvature tensor V are given by [8]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y]. \tag{15}$$

$$P(\xi, Y)Z = -\{g(Y, Z) + \frac{1}{n-1} S(Y, Z)\}\xi, \tag{16}$$

$$P(X, Y)\xi = 0. \tag{17}$$

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}\{ g(Y, Z)X - g(X, Z)Y \}. \tag{18}$$

$$\begin{aligned} V(X, Y)Z &\\ = R(X, Y)Z &- \frac{1}{n-2}\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\} \\ &+ \frac{r}{(n-1)(n-2)}\{g(Y, Z)X - g(X, Z)Y\}. \end{aligned} \tag{19}$$

3. W_2 – Curvature Tensor of a Kenmotsu Manifold

Pokhariyal and Mishra [10] have defined a new curvature tensor $'W_2$ as

$$'W_2(X, Y, Z, U) = 'R(X, Y, Z, U) + \frac{1}{n-1}[g(X, Z)S(Y, U) - g(Y, Z)S(X, U)], \tag{20}$$

for arbitrary vector fields X, Y, Z and U, where S is the Ricci tensor of type (0, 2) and $'W_2(X, Y, Z, U) = g(W_2(X, Y)Z, U)$ and $'R(X, Y, Z, U) = g(R(X, Y)Z, U)$ called W_2-curvature tensor of M^n.

Proposition: *On an n- dimensional Kenmotsu manifold M^n,*

$\eta(W_2(X, Y)Z) = 0$.

Proof: From equation (20), we have

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{n-1}[g(X, Z)QY - g(Y, Z)QX]. \tag{21}$$

Taking the inner product of above equation with ξ and using equations (10), (11) and (12), we get

$$\eta(W_2(X, Y)Z) = 0. \tag{22}$$

Theorem 1: *On an n- dimensional Kenmotsu manifold M^n,*

$R(\xi, X).W_2 = 0$ if and only if $W_2 = 0$.

Proof: Let on an n- dimensional Kenmotsu manifold $R(\xi, X).W_2 = 0$, then

$$R(\xi, X)W_2(Y, Z)U - W_2(R(\xi, X)Y, Z)U - W_2(Y, R(\xi, X)Z)U - W_2(Y, Z)R(\xi, X)U = 0. \tag{23}$$

From equations (9) and (23), we have

$$\begin{aligned}&\eta(W_2(Y, Z)U)X - 'W_2(Y, Z, U, X)\xi - \eta(Y)W_2(X, Z)U\\ &+ g(X, Y)W_2(\xi, Z)U - \eta(Z)W_2(Y, X)U + g(X, Z)W_2(Y, \xi)U\\ &- \eta(U)W_2(Y, Z)X + \mathbf{g(X, U)W_2(Y, Z)\xi} = 0.\end{aligned} \tag{24}$$

Taking the inner product of above equation with ξ, we get

$$\eta(W_2(Y, Z)U)\,\eta(X) - {}'W_2(Y, Z, U, X) - \eta(Y)\,\eta(W_2(X, Z)U)$$
$$+ g(X, Y)\,\eta(W_2(\xi, Z)U) - \eta(Z)\,\eta(W_2(Y, X)U)$$
$$+ g(X, Z)\,\eta(W_2(Y, \xi)U) - \eta(U)\,\eta(W_2(Y, Z)X) + \mathbf{g(X, U)\,\eta(W_2(Y, Z)\xi)} = 0, \tag{25}$$

which on using equation (22) gives

$${}'W_2(Y, Z, U, X) = 0,$$

i.e. $W_2 = 0$.

Conversely, suppose $W_2 = 0$, then from equation (23), we have

$$R(\xi, X).W_2 = 0.$$

This completes the proof.

Theorem 2: *An n-dimensional Kenmotsu manifold M^n satisfying $W_2(\xi, X).R = 0$, is an Einstein manifold.*

Proof: Let $W_2(\xi, X).R = 0$, then we have

$$W_2(\xi, X).R(Y, Z)U - R(W_2(\xi, X)Y, Z)U - R(Y, W_2(\xi, X)Z)U$$
$$\mathbf{- R(Y, Z)\,W_2(\xi, X)U} = 0. \tag{26}$$

Now putting $X = \xi$ in equation (21) and using equations (9) and (12), we obtain

$$W_2(\xi, Y)Z = \eta(Z)[Y + \frac{1}{n-1}QY]. \tag{27}$$

Now from equations (26) and (27), we have

$$\eta(R(Y,Z)U)\{X + \frac{1}{n-1}QX\} - \eta(Y)\{R(X, Z)U + \frac{1}{n-1}R(QX, Z)U\}$$
$$- \eta(Z)\{R(Y, X)U + \frac{1}{n-1}R(Y, QX)U\} - \eta(U)\{R(Y, Z)X + \frac{1}{n-1}R(Y, Z)QX\}$$
$$= 0, \tag{28}$$

which on taking the inner product with ξ and using equations (4), (11) and (12) gives

$$\mathbf{-\,\eta(U)\{\,\eta(Z)g(X, Y) - \eta(Y)g(X, Z)\} - \frac{1}{n-1}\{\eta(Z)S(X, Y) - \eta(Y)S(X, Z)\}\eta(U) = 0.} \tag{29}$$

Putting $U = Z = \xi$ in above equation and using equations (4) and (11), we get

$$\mathbf{S(X, Y) = (1\text{-}n)\, g(X, Y),} \tag{30}$$

which shows that M^n is an Einstein Manifold.

Theorem 3: *An n-dimensional Kenmotsu manifold M^n satisfying $P(\xi, X).W_2 = 0$, is an Einstein manifold.*

Proof: Let $P(\xi, X).W_2 = 0$, then

$$P(\xi, X)W_2(Y,Z)U - W_2(P(\xi, X)Y, Z)U - W_2(Y, P(\xi, X)Z)U$$
$$- W_2(Y,Z)P(\xi, X)U = 0. \tag{31}$$

Using equations (11) and (16) in above equation, we have

$$\mathbf{-'W_2(Y, Z, U, X)\xi - \frac{1}{n-1}\,'W_2(Y, Z, U, QX)\xi}$$
$$+ \{g(X, Y) + \frac{1}{n-1}S(X, Y)\}W_2(\xi, Z)U$$
$$+ \{g(X, Z) + \frac{1}{n-1}S(X, Z)\}W_2(Y, \xi)U$$
$$+\{g(X, U) + \frac{1}{n-1}S(X, U)\}W_2(Y, Z)\xi = 0. \tag{32}$$

Now taking the inner product of above equation with ξ and using equation (22), we get

$$'W_2(Y, Z, U, QX) = (1\text{-}n)\ 'W_2(Y, Z, U, X),$$

which on using equation (21), gives

$$g(QX, R(Y, Z)U) + \frac{1}{n-1}\{g(Y, U)g(QX, QZ) - g(Z, U)g(QX, QY)\}$$
$$= (1\text{-}n)[\, g(X, R(Y, Z)U) + \frac{1}{n-1}\{g(Y, U)g(X, QZ) - g(Z, U)g(X, QY)\}].$$

Putting $Z = U = \xi$ in above equation and using equations (4), (8) and (11), we get

$$-S(X, QY) = 2(n\text{-}1)S(X, Y) + (n\text{-}1)^2 g(X, Y),$$

which on using equation (11), gives

$$S(X, Y) = (1-n)g(X, Y).$$

This completes the proof.

Theorem 4: ***An n-dimensional Kenmotsu manifold M^n satisfying $W_2(\xi, X).P = 0$, is an Einstein manifold.***

Proof: Let $W_2(\xi, X).P = 0$, then we have

$$W_2(\xi, X)P(Y, Z)U - P(W_2(\xi, X)Y, Z)U - P(Y, W_2(\xi, X)Z)U$$
$$- P(Y, Z)W_2(\xi, X)U = 0, \tag{33}$$

which on using equation (27), gives

$$\eta(P(Y, Z)U)\{X + \frac{1}{n-1}QX\} - \eta(Y)\{P(X, Z)U + \frac{1}{n-1}P(QX, Z)U\}$$

$$- \eta(Z)\{ P(Y, X)U + \frac{1}{n-1} P(Y, QX)U\} - \eta(U)\{P(Y, Z)X + \frac{1}{n-1}P(Y, Z)QX\} = 0. \tag{34}$$

Now taking the inner product of above equation with ξ and using equation (11), we get

$$\eta(Y)\{ \eta(P(X, Z)U) + \frac{1}{n-1} \eta (P(QX, Z)U)\}$$

$$+ \eta(Z)\{ \eta(P(Y, X)U) + \frac{1}{n-1} \eta(P(Y, QX)U)\}$$

$$+ \eta(U)\{ \eta(P(Y, Z)X) + \frac{1}{n-1} \eta(P(Y, Z)QX)\} = 0. \tag{35}$$

Putting $U = Z = \xi$ in above equation and using equations (15), (16) and (17), we get

$$S(QX, Y) = (n-1)^2g(X, Y),$$

which on using equation (11), gives

$$\mathbf{S(X, Y) = (1-n)g(X, Y).}$$

This completes the proof.

Theorem 5: *The W_2-curvature tensor and projective curvature tensor of the Riemannian manifold M^n are linearly dependent if and only if M^n is an Einstein Manifold.*

Proof: Let

$$W_2(X, Y)Z = \alpha P(X, Y)Z,$$

where α being any non-zero constant. In view of equations (15) and (21), above equation assumes the form

$$(1-\alpha) R(X, Y)Z + \frac{1}{n-1}\{g(X, Z)QY - g(Y, Z)QX\} + \frac{\alpha}{n-1}\{S(Y, Z)X - S(X, Z)Y\} = 0,$$

which can be written as

$$(1-\alpha)\, 'R(X, Y, Z, U) + \frac{1}{n-1}\{g(X, Z)S(Y, U) - g(Y, Z)S(X, U)\}$$
$$+ \frac{\alpha}{n-1}\{S(Y, Z)g(X, U) - S(X, Z)g(Y, U)\} = 0.$$

Now putting $X = U = e_i$ in above equation and taking summation over i, $1 \leq i \leq n$, we get

$$(1-\alpha) S(Y, Z) + \frac{1}{n-1}\{S(Y, Z) - r\, g(Y, Z)\} + \frac{\alpha}{n-1}\{n\, S(Y, Z) - S(Y, Z)\} = 0,$$

i.e. $$S(Y, Z) = \frac{r}{n} g(Y, Z) \Rightarrow QY = \frac{r}{n} Y,$$

which shows that M^n is an Einstein manifold.

Conversely, let M^n be an Einstein manifold, i.e. $S(Y, Z) = \frac{r}{n} g(Y, Z)$ and $QY = \frac{r}{n} Y$, then from equations (15) and (21), we have

$$W_2(X, Y)Z = \alpha P(X, Y)Z.$$

Theorem 6: *A necessary and sufficient condition for a Riemannian manifold M^n to be an Einstein manifold is that the W_2-curvature tensor and concircular curvature tensor C are linearly dependent.*

Proof: Let $W_2(X, Y)Z = \alpha C(X, Y)Z$,

where α being any non-zero constant. In consequence of equations (18) and (21), we have

$$(1-\alpha) R(X, Y)Z + \frac{1}{n-1}\{ g(X, Z)QY - g(Y, Z)QX \}$$

$$+ \frac{\alpha r}{n(n-1)} \{ g(Y, Z)X - g(X, Z)Y \} = 0,$$

from which, we get

$$(1-\alpha)\ 'R(X, Y, Z, U) + \frac{1}{n-1} \{ g(X, Z)S(Y, U) - g(Y, Z)S(X, U) \}$$
$$+ \frac{\alpha r}{n(n-1)} \{ g(Y, Z)g(X, U) - g(X, Z)g(Y, U) \} = 0.$$

Now putting $X = U = e_i$ in above equation and taking the summation over i, $1 \leq i \leq n$, we get

$$\frac{(n-1)(1-\alpha)+1}{(n-1)} S(Y, Z) + \frac{\{(n-1)\alpha - n\} r}{n(n-1)} g(Y, Z) = 0,$$

which can be written as

$$S(Y, Z) = \frac{k}{n} g(Y, Z),$$

where $k = \frac{\{(n-1)\alpha - n\} r}{\{(n-1)(\alpha-1)+1\}}$. This shows that Riemannian manifold is an Einstein manifold. Converse part is obvious from the equations (18) and (21).

Theorem 7: *A Riemannian manifold M^n becomes an Einstein manifold if and only if conformal curvature tensor and W_2-curvature tensor of the manifold are linearly dependent.*

Proof: Let $W_2(X, Y)Z = \alpha\, V(X, Y)Z$.

The above equation on straight forward calculations, gives

$$(1-\alpha)\ 'R(X, Y, Z, U) + \frac{1}{n-1} \{g(X, Z)\, S(Y, U) - g(Y, Z)\, S(X, U)\}$$
$$+ \frac{\alpha}{(n-2)} \{ S(Y, Z)\, g(X, U) - S(X, Z)\, g(Y, U) + g(Y, Z)S(X, U)$$
$$- g(X, Z)S(Y, U)\} - \frac{\alpha r}{(n-1)(n-2)} \{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\} = 0.$$

Now putting $X = U = e_i$ in above equation and taking summation over i, $1 \leq i \leq n$, we get

$$(1-\alpha)\,S(Y,Z) + \frac{1}{n-1}\{S(Y,Z) - r\,g(Y,Z)\}$$
$$+\frac{\alpha}{(n-2)}\{n\,S(Y,Z) - S(Y,Z) + r\,g(Y,Z) - S(Y,Z)\}$$
$$-\frac{\alpha\, r}{(n-1)(n-2)}\{n\,g(Y,Z) - g(Y,Z)\} = 0,$$

which reduces to

$$S(Y,Z) = \frac{r}{n}\,g(Y,Z) \Rightarrow QY = \frac{r}{n}Y.$$

This shows that Riemannian manifold is an Einstein manifold. Converse part is obvious from equations (19) and (21).

Corollary: *In an n-dimensional Riemannian manifold M^n, the following statements are equivalent-*

(i) M^n is an Einstein manifold,

(ii) W_2- curvature tensor and projective curvatures are linearly dependent,

(iii) W_2- curvature tensor and concircular curvature tensors are linearly dependent,

(iv) W_2- curvature tensor and conformal curvature tensors are linearly dependent.

4. The Irrotational W_2-Curvature Tensor

Definition: Let ∇ be a Riemannian connection. The rotation (Curl) of W_2-curvature tensor on Riemannian manifold M^n is defined as

$$\begin{aligned} \mathrm{Rot}W_2 = (\nabla_U W_2)(X, Y)Z + (\nabla_X W_2)(U, Y)Z \\ + (\nabla_Y W_2)(X, U)Z - (\nabla_Z W_2)(X, Y)U. \end{aligned} \tag{36}$$

In consequence of Bianchi's second identity for Riemannian connection ∇, equation (36) becomes

$$\mathrm{Rot}W_2 = -(\nabla_Z W_2)(X, Y)U. \tag{37}$$

If the W_2-curvature tensor is irrotational, then curl $W_2 = 0$ and hence

$$(\nabla_Z W_2)(X, Y)U = 0,$$

which gives

$$\nabla_Z (W_2(X, Y)U) = W_2(\nabla_Z X, Y)U + W_2(X, \nabla_Z Y)U + W_2(X, Y)\nabla_Z U. \quad (38)$$

Theorem 8: *The W_2-curvature tensor in a Kenmotsu manifold M^n is irrotational if and only if*

$$R(X, Y)Z = g(X, Z)Y - g(Y, Z)X + \frac{1}{n-1}\{ \eta(X)(\nabla_Z Q)(Y) - \eta(Y)(\nabla_Z Q)(X) \}.$$

In particular, if $\eta(X)(\nabla_Z Q)(Y) = \eta(Y)(\nabla_Z Q)(X)$, then the manifold is locally isometric to the hyperbolic space $H^n(-1)$.

Proof: Let W_2-curvature tensor in M^n be irrotational then putting $U = \xi$ in equation (38), we get

$$\nabla_Z (W_2(X, Y)\xi) = W_2(\nabla_Z X, Y)\xi + W_2(X, \nabla_Z Y)\xi + W_2(X, Y)\nabla_Z \xi. \quad (39)$$

Putting $Z = \xi$ in equation (21) and using equations (4) and (8), we get

$$W_2(X, Y)\xi = \{ \eta(X)Y - \eta(Y)X\} + \frac{1}{n-1}\{ \eta(X)QY - \eta(Y)QX \}. \quad (40)$$

Using above equation in equation (39), we obtain

$$(\nabla_Z \eta)(X)Y - (\nabla_Z \eta)(Y)X + \frac{1}{n-1}\{(\nabla_Z \eta)(X)QY - (\nabla_Z \eta)(Y)QX$$

$$+ \eta(X)(\nabla_Z Q)(Y) - \eta(Y)(\nabla_Z Q)(X) \}$$

$$= W_2(X, Y)Z - \eta(Z)[\{ \eta(X)Y - \eta(Y)X\} + \frac{1}{n-1}\{ \eta(X)QY - \eta(Y)QX \}], \quad (41)$$

which on using equation (7) gives

$$g(X, Z)Y - g(Y, Z)X + \frac{1}{n-1}\{g(X, Z)QY - g(Y, Z)QX + \eta(X)(\nabla_Z Q)(Y)$$

$$- \eta(Y)(\nabla_Z Q)(X) \} = W_2(X, Y)Z. \quad (42)$$

Using equation (21) in above equation, we have

$$R(X, Y)Z = g(X, Z)Y - g(Y, Z)X + \frac{1}{n-1}\{ \eta(X)(\nabla_Z Q)(Y) - \eta(Y)(\nabla_Z Q)(X) \}. \quad (43)$$

Conversely, retreating the steps, we can show that W_2-curvature tensor is an irrotational.

Now if $\eta(X)(\nabla_Z Q)(Y) = \eta(Y)(\nabla_Z Q)(X)$, then equation (43) reduces to

$$R(X, Y)Z = -(g(Y, Z)X - g(X, Z)Y),$$

which shows that Kenmotsu manifold is locally isometric to hyperbolic space $H^n(-1)$.

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A sufficient condition for the existence of 2-repeated low-density burst error correcting code *

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Abstract

This paper presents upper bound on the number of parity-check digits required for linear codes that correct 2-repeated low-density burst errors of length b(fixed) with weight w or less ($w \leq b$). Further, an illustration for the existence of a linear code that corrects 2-repeated burst errors of length 3(fixed) with weight 2 or less over GF(2) has also been provided.

Keywords and Phrases: *Error correcting code, Burst error, Low-density burst error, Repeated low-density burst error, CT burst.*

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1. Introduction

In the theory of error correcting codes, burst errors have played a dominant role amongst the several kinds of errors (refer Abramson (1959), Fire (1959)). A burst of length b is defined as follows:

Definition 1. A burst of length b is a vector whose only non-zero components are among some b consecutive components, the first and the last of which are non-zero.

It was observed by Chien and Tang (1965) that in many channels errors occur in the form of a burst but do not occur towards the end digits of the burst. They modified the definition as follows:

Definition 2. A burst of length b is a vector whose only non-zero components are confined to some b consecutive positions, the first of which is non-zero.

There are cetain channels viz. studied by Alexander, Gryb and Nast (1960) which deals with such bursts commonly known as CT bursts.

A further modification to this definition was made by Dass (1980) which is useful for channels not producing errors near the end of a code word and is as follows:

Definition 3. A burst of length b(fixed) is an n-tuple whose only non-zero components are confined to b consecutive positions, the first of which is non-zero and the number of its starting positions in an n-tuple is the first $n-b+1$ positions.

In situations like lightning or other similar disturbances which introduce burst errors usually operate in a way that, over a given length, some digits are received correctly while others get corrupted i.e. errors occur in the form of low-density bursts. The study of low-density bursts was initiated by Wyner (1963). Further study on low-density burst error correcting codes has been made by Sharma and Dass (1974), Dass (1975), Dass (1983) and others.

As has been observed that in very busy communication channels errors repeat themselves, Dass and Garg (2009) studied 2-repeated burst errors of length b(fixed) and this concept was generalized for m-repeated bursts of length b(fixed) by Dass, Garg and Zannetti (2008a). Further results have also been obtained by Dass, Garg and Zannetti (2008b).

Different situations demand development of codes which correct those errors that are repeated burst errors of specified length, i.e., repeated low-density

burst errors of length b(fixed) with weight w or less. Such 2-repeated low-density burst errors and also the general case of m-repeated bursts have been studied by Dass and Garg (2011). An m-repeated low-density burst of length b(fixed) with weight w or less ($w \leq b$) has been defined as follows:

Definition 4. An m-repeated low-density burst of length b(fixed) with weight w or less is an n-tuple whose only non-zero components are confined to m distinct sets of b consecutive positions, the first component of each set is non-zero where each set can have at the most w non-zero components ($w \leq b$), and the number of its starting positions in an n-tuple is among the first $n-mb+1$ positions.

In particular, 2-repeated low-density burst error of length b(fixed) with weight w or less becomes as follows:

Definition 5. A 2-repeated low-density burst of length b(fixed) with weight w or less is an n-tuple whose only non-zero components are confined to two distinct sets of b consecutive positions, the first component of each set is non-zero where each set can have at the most w non-zero components ($w \leq b$), and the number of its starting positions in an n-tuple is among the first $n-2b+1$ positions.

It may be noted that according to Definition 5, when the first low-density burst of length b(fixed) with weight w or less starts from the first position of the vector then the second low-density burst of length b(fixed) with weight w or less is in the last $n-b$ components. When the first low-density burst of length b(fixed) with weight w or less starts from the second position of the vector then the second low-density burst of length b(fixed) with weight w or less will be in the last $n-b-1$ components. In general, when the first low-density burst of length b(fixed) with weight w or less starts from the i-th position, then the second low-density burst of length b(fixed) with weight w or less will be in the remaining last $(n-b-i+1)$ components where i can take the values from 1 to $n-2b+1$ since the starting positions are among the first $n-2b+1$ components. Further, in the last $2b-1$ components only single low-density burst of length b(fixed) with weight w or less can exist, with the starting positions to be atmost upto $n-b+1$.

Lower bound for the correction of m-repeated low-density bursts of length b(fixed) with weight w or less ($w \leq b$) has been obtained by Dass and Garg (2012). In the same paper, an upper bound for codes which can detect such errors has also been obtained.

This paper has been organized as follows:

In section 2, an upper bound for the existence of linear codes that can correct any 2-repeated low-density burst errors of length b(fixed) with weight w or less is given. The paper concludes with an illustration of such a code.

In what follows a linear code will be considered as a subspace of the space of all n-tuples over $GF(q)$. The distance between two vectors shall be considered in the Hamming sense.

2. Bound for codes correcting 2-repeated low-density burst errors

In this section, we derive an upper bound on the number of parity-check digits that assures the existence of a code capable of correcting 2-repeated low-density burst errors of length b(fixed) with weight w or less. The proof involves a technique given by Dass (1983) which is a suitable modification of the technique used by Sacks (1958) in establishing the well-known Varsharmov-Gilbert bound. Before deriving the main result, we state below a result obtained by Dass and Garg (2012, Theorem 1) to be used.

Result. An (n, k) linear code over GF(q) that corrects m-repeated low-density bursts of length b(fixed) with weight w or less ($w \leq b$) must satisfy:

$$q^{n-k} \geq \sum_{i=0}^{m} \binom{n-ib+i}{i} (q-1)^i [1+(q-1)]^{i(b-1,w-1)},$$

where $[1+x]^{(m,r)} = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \ldots + \binom{m}{r}x^r.$

We now derive the following theorem:

Theorem 1. *Given positive integers w and b such that $w \leq b$, there exists an (n,k) linear code that corrects all 2-repeated low-density burst errors of length b(fixed) $(n > 4b)$ with weight w or less provided that*

$$
\begin{aligned}
q^{n-k} \quad > \quad & \left([1+(q-1)]^{(b-1,w-1)}\right) \\
& \times \Bigg\{ \sum_{i=0}^{3} \binom{n-(i+1)b+i}{i} (q-1)^i [1+(q-1)]^{i(b-1,w-1)} \\
& + \sum_{i=1}^{n-4b+1} \Bigg\{ \sum_{k_2=1}^{b-1} \{ L_1(b,k_2,r_4,r_5,r_6) \\
& \times \{(n-4b+k_2-i+2)(q-1)[1+(q-1)]^{(b-1,w-1)}\}\} \Bigg\} \\
& + \sum_{\substack{i=n-4b+1+k_3 \\ 1\leq k_3\leq b-1}} \Bigg\{ \sum_{k_2=k_3}^{b-1} \{ L_1(b,k_2,r_4,r_5,r_6) \\
& \times \{(n-4b+k_2-i+2)(q-1)[1+(q-1)]^{(b-1,w-1)}\}\} \Bigg\} \\
& + \sum_{i=1}^{n-4b+1} \Bigg\{ \bigg((q-1)[1+(q-1)]^{(b-1,w-1)} \bigg) \\
& \times \Bigg\{ (n-4b-i+2) \sum_{k_2=1}^{b-1} L_1(b,k_2,r_4,r_5,r_6) \\
& \\
& + \sum_{\substack{k_2=k_3 \\ 1\leq k_3\leq b-1}}^{b-1} L_1(b,k_2,r_4,r_5,r_6) \Bigg\} \Bigg\} \\
& + \Bigg\{ \left((q-1)[1+(q-1)]^{(b-1,w-1)}\right) \sum_{\substack{k_2=k_4 \\ 1\leq k_4\leq b-1}}^{b-1} L_1(b,k_2,r_4,r_5,r_6) \Bigg\} \\
& + (n-3b+1) \sum_{k_2=1}^{b-1} L_1(b,k_2,r_4,r_5,r_6) + \sum_{\substack{k_2=k_4 \\ 1\leq k_4\leq b-1}}^{b-1} L_1(b,k_2,r_4,r_5,r_6) \\
& + (n-4b+3) \sum_{\substack{k_5,k_6 \\ 1\leq k_6\leq b-1 \\ 1\leq k_5\leq b-k_6}} L_3(b,k_5,k_6,r_7,r_8,r_9,r_{10},r_{11}) \\
& + \sum_{\substack{i_1,k_5,k_6 \\ 1\leq i_1\leq b-2 \\ i_1+1\leq k_6\leq b-1 \\ 1\leq k_5\leq b-k_6}} L_3(b,k_5,k_6,r_7,r_8,r_9,r_{10},r_{11})
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{\substack{i_1,k_5,k_6\\ 1\le i_1\le b-2\\ 1\le k_6\le b-1\\ i_1+1\le k_5\le b-k_6}} L_3(b,k_5,k_6,r_7,r_8,r_9,r_{10},r_{11})\Bigg\}\\
&+\sum_{k_1=1}^{b-1}\Bigg\{L(b,k_1,r_1,r_2,r_3)\Bigg\{\sum_{i=0}^{2}\binom{n-(2+i)b+(k_1+i)}{i}\\
&\times(q-1)^i[1+(q-1)]^{i(b-1,w-1)}+(n-4b+k_1+1)\\
&\times\sum_{k_2=1}^{b-1}L_1(b,k_2,r_4,r_5,r_6)+\sum_{\substack{k_2=k_3\\ 1\le k_3\le b-1}}^{b-1}L_1(b,k_2,r_4,r_5,r_6)\Bigg\}\Bigg\}\\
&+\sum_{\substack{k_5,k_6\\ 1\le k_6\le b-1\\ 1\le k_5\le b-k_6}} L_2(b,k_5,k_6,r_7,r_8,r_9,r_{10},r_{11})\\
&\times\big\{1+(n-4b+k_5+k_6+1)(q-1)[1+(q-1)]^{(b-1,w-1)}\big\}\\
&+\sum_{\substack{k_8,k_9,k_{10}\\ 1\le k_{10}\le b-1\\ 1\le k_9\le b-k_{10}\\ 1\le k_8\le b-k_9}} L_4(b,k_8,k_9,k_{10},r_{12},r_{13},r_{14},r_{15},r_{16},r_{17},r_{18}). \qquad (1)
\end{aligned}
$$

where

$$[1+x]^{(m,r)}=1+\binom{m}{1}x+\binom{m}{2}x^2+\ldots+\binom{m}{r}x^r,$$

$$
\begin{aligned}
&L(b,k_1,r_1,r_2,r_3)\\
&\quad=\sum_{r_1,r_2,r_3}\binom{b-k_1}{r_1}\binom{k_1-1}{r_2}\binom{b-k_1-1}{r_3}(q-1)^{r_1+r_2+r_3+1},\\
&L_1(b,k_2,r_4,r_5,r_6)\\
&\quad=\sum_{r_4,r_5,r_6}\binom{b-k_2}{r_4}\binom{k_2-1}{r_5}\binom{b-k_2-1}{r_6}(q-1)^{r_4+r_5+r_6+2},\\
&L_2(b,k_5,k_6,r_7,r_8,r_9,r_{10},r_{11})\\
&\quad=\sum_{\substack{r_7,r_8,r_9,\\ r_{10},r_{11}}}\binom{b-k_5}{r_7}\binom{k_5-1}{r_8}\binom{b-k_5-k_6}{r_9}\binom{k_6-1}{r_{10}}\\
&\quad\times\binom{b-k_6-1}{r_{11}}(q-1)^{r_7+r_8+r_9+r_{10}+r_{11}+2},
\end{aligned}
$$

$$
\begin{aligned}
&L_3(b,k_5,k_6,r_7,r_8,r_9,r_{10},r_{11})\\
&\quad=\sum_{\substack{r_7,r_8,r_9,\\ r_{10},r_{11}}}\binom{b-k_5}{r_7}\binom{k_5-1}{r_8}\binom{b-k_5-k_6}{r_9}\binom{k_6-1}{r_{10}}\\
&\quad\times\binom{b-k_6-1}{r_{11}}(q-1)^{r_7+r_8+r_9+r_{10}+r_{11}+3},\\
&L_4(b,k_8,k_9,k_{10},r_{12},r_{13},r_{14},r_{15},r_{16},r_{17},r_{18})\\
&\quad=\sum_{r_{12},\dots,r_{18}}\binom{b-k_8}{r_{12}}\binom{k_8-1}{r_{13}}\binom{b-k_8-k_9}{r_{14}}\binom{k_9-1}{r_{15}}\binom{b-k_9-k_{10}}{r_{16}}\\
&\quad\times\binom{k_{10}-1}{r_{17}}\binom{b-k_{10}-1}{r_{18}}(q-1)^{r_{12}+r_{13}+r_{14}+r_{15}+r_{16}+r_{17}+r_{18}+3}
\end{aligned}
$$

with

$$
\begin{aligned}
&0\le r_1\le w-1, 0\le r_2\le 2w-2, 0\le r_3\le w-1,\\
&r_2+r_3\ge w-1, r_1+r_2+r_3\le 2w-2,\\
&0\le r_4\le w-1, 0\le r_5\le 2w-2, 0\le r_6\le w-1,\\
&r_5+r_6\ge w-1, r_4+r_5+r_6\le 2w-2,\\
&0\le r_7\le w-1, 0\le r_8\le 2w-2, 0\le r_9\le w-1, 0\le r_{10}\le 2w-2,\\
&0\le r_{11}\le w-1, r_{10}+r_{11}\ge w-1, r_8+r_9+r_{10}+r_{11}\ge 2w-2,\\
&r_7+r_8+r_9+r_{10}+r_{11}\le 3w-3,\\
&0\le r_{12}\le w-1, 0\le r_{13}\le 2w-2, 0\le r_{14}\le w-1,\\
&0\le r_{15}\le 2w-2, 0\le r_{16}\le w-1, 0\le r_{17}\le 2w-2, 0\le r_{18}\le w-1,\\
&r_{17}+r_{18}\ge w-1,\\
&r_{15}+r_{16}+r_{17}+r_{18}\ge 2w-2,\\
&r_{13}+r_{14}+r_{15}+r_{16}+r_{17}+r_{18}\ge 3w-3,\\
&r_{12}+r_{13}+r_{14}+r_{15}+r_{16}+r_{17}+r_{18}\le 4w-4\,.
\end{aligned}
$$

Proof. The existence of such a code will be shown by constructing an appropriate $(n-k)\times n$ parity-check matrix H. Firstly, we construct a matrix H'

from which the requisite parity-check matrix H shall be obtained by reversing the order of its columns altogether. Any non-zero $(n-k)$-tuple is chosen as the first column h_1 of H'. Subsequent columns are added to H' such that after having selected the first $j-1$ columns $h_1, h_2, \ldots, h_{j-1}$, the jth column h_j may be added provided that it is not a linear combination of any $w-1$ or fewer columns from amongst the immediately preceding $b-1$ columns and w or fewer columns from amongst any b consecutive columns from the first $j-b$ columns, together with any two sets of w or fewer columns, each chosen from a distinct set of b consecutive columns from amongst all the $j-1$ columns (note that b consecutive columns here do not include less than b columns).

In other words, h_j may be added provided that

$$\begin{aligned} h_j \neq\ & \alpha_{j_1}h_{j_1} + \alpha_{j_2}h_{j_2} + \ldots + \alpha_{j_{w-1}}h_{j_{w-1}} + \beta_{r_1}h_{r_1} + \beta_{r_2}h_{r_2} + \ldots + \beta_{r_w}h_{r_w} \\ & + \gamma_{\ell_1}h_{\ell_1} + \gamma_{\ell_2}h_{\ell_2} + \ldots + \gamma_{\ell_w}h_{\ell_w} + \delta_{p_1}h_{p_1} + \delta_{p_2}h_{p_2} + \ldots + \delta_{p_w}h_{p_w} \quad (2) \end{aligned}$$

where the $h_{j_1}, h_{j_2}, \ldots, h_{j_{w-1}}$ are any $w-1$ columns among $h_{j-(b-1)}, h_{j-(b-2)}, \ldots, h_{j-1}$ and h_r, h_ℓ, h_p are any w columns each from three sets of b consecutive columns such that one of the sets of b consecutive columns is amongst the first $j-b$ columns, and the other two sets of b consecutive columns are distinct and from amongst all the $j-1$ columns.

It may be noted that either all β_r, γ_ℓ, δ_p are zero or if β_t is the last non-zero coefficient of the first set of b consecutive columns, then

$$b \leq t \leq j-b, \alpha_j, \beta_r, \gamma_\ell, \delta_p \text{ are in GF}(q). \qquad (3)$$

Also if γ_{t_1}, is the last non-zero coefficient of the second set of b consecutive columns, then

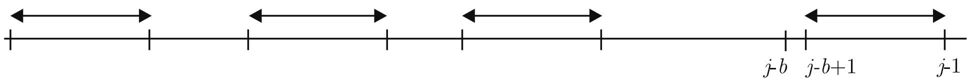
$$b \leq t_1 \leq j-1\,. \qquad (4)$$

The restriction $t_1 \leq j-1$ is obviously satisfied since the selection of the columns is out of all the $j-1$ previously chosen columns.

The condition (2) ensures that there would not be a code word which is expressible as sum or difference of two vectors, each of which is a 2-repeated low-density burst of length b(fixed) with weight w or less.

To enumerate all possible linear combinations on the R.H.S of (2), there are as many as 9 different cases to be examined, We analyze these as follows:

Case 1. *When h_j are selected from $h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}$ and the h_r, h_ℓ, h_p are selected from three distinct sets of b consecutive columns from amongst the first $j-b$ columns.*



In this case, the number of ways in which coefficients α_j can be selected is

$$[1+(q-1)]^{(b-1,w-1)}. \tag{5}$$

The number of ways in which the β_r, γ_ℓ, δ_p can be selected is equivalent to enumerate the number of 3-repeated low-density bursts of length b(fixed) with weight w or less in a vector of length $j-b$, which is (refer Theorem 1, Dass and Garg (2012))

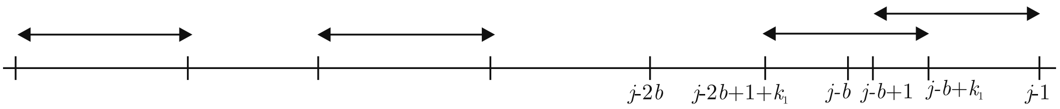
$$\sum_{i=0}^{3} \binom{(j-b)-ib+i}{i} (q-1)^i [1+(q-1)]^{i(b-1,w-1)}. \tag{6}$$

Therefore, the total number of choices of coefficients in this case is

$$[1+(q-1)]^{(b-1,w-1)} \left\{ \sum_{i=0}^{3} \binom{j-(i+1)b+i}{i} (q-1)^i [1+(q-1)]^{i(b-1,w-1)} \right\}. \tag{7}$$

Case 2. *When the h_p are selected from $h_{j-2b+2}, \ldots, h_{j-1}$ such that all the h_p are neither taken from $h_{j-2b+2}, \ldots, h_{j-b}$ nor from $h_{j-b+1}, \ldots, h_{j-1}$ i.e. the last*

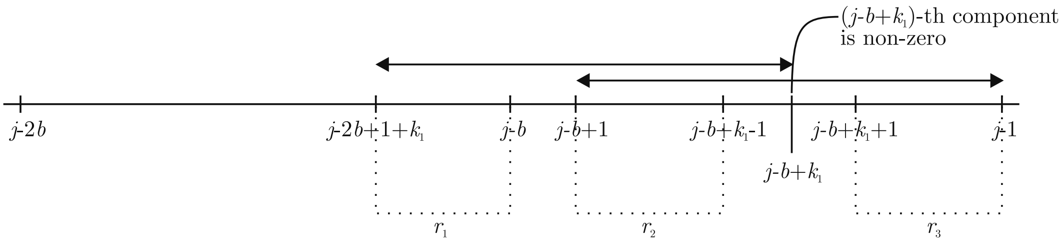
h_p is among $h_{j-b+1}, \ldots, h_{j-1}$. The h_r and h_ℓ are selected from two distinct sets of b consecutive columns among the first $j-2b+k_1$ columns, $1 \leq k_1 \leq b-1$.



In this case, the coefficients δ_p are selected from b consecutive components as w or less non-zero components, which starts from the $(j-2b+1+k_1)$-th component which may obviously continue upto $(j-b+k_1)$-th component. We shall first select $w-1$ or less non-zero components among $(j-2b+1+k_1, \ldots, j-b+k_1-1)$-th positions, the $(j-b+k_1)$-th component is non-zero, together with $w-1$ or less non-zero components among $(j-b+1, \ldots, j-1)$-th positions. Now the aim is to select the coefficients β_r and γ_ℓ which are w or less non-zero components each selected from a distinct set of b consecutive components among the first $j-2b+k_1$ positions, $1 \leq k_1 \leq b-1$.

In order to do so, let us choose r_1 components from the $(j-2b+k_1+1, \ldots, j-b)$-th positions, r_2 components from the $(j-b+1, \ldots, j-b+k_1-1)$-th positions and r_3 components from the $(j-b+k_1+1, \ldots, j-1)$-th positions, where

$$0 \leq r_1 \leq w-1, \quad 0 \leq r_2 \leq 2w-2, \quad 0 \leq r_3 \leq w-1. \tag{8}$$



Keeping in view the situations considered in case 1, r_1, r_2, r_3 should be such that

$$r_2 + r_3 \geq w - 1, r_1 + r_2 + r_3 \leq 2w - 2. \tag{9}$$

Such a selection of coefficients gives us

$$\sum_{r_1,r_2,r_3} \binom{b-k_1}{r_1}\binom{k_1-1}{r_2}\binom{b-k_1-1}{r_3}(q-1)^{r_1+r_2+r_3}$$

possible linear combinations where r_1, r_2, r_3 each satisfy the constraints stated in (8) and (9). Also, the $(j-b+k_1)$-th component can be selected in $(q-1)$ ways, therefore selection of coefficients give us

$$\sum_{r_1,r_2,r_3} \binom{b-k_1}{r_1}\binom{k_1-1}{r_2}\binom{b-k_1-1}{r_3}(q-1)^{r_1+r_2+r_3+1} \tag{10}$$

choices.

Suppose $L(b, k_1, r_1, r_2, r_3)$ represents the expression in (10) with conditions in (8) and (9). Now to select β_r and γ_ℓ, it is equivalent to enumerate the number of 2-repeated low-density bursts of length b(fixed) with weight w or less in a vector of length $j-2b+k_1$, which gives us (refer Theorem 1 for m =2, Dass and Garg (2012))

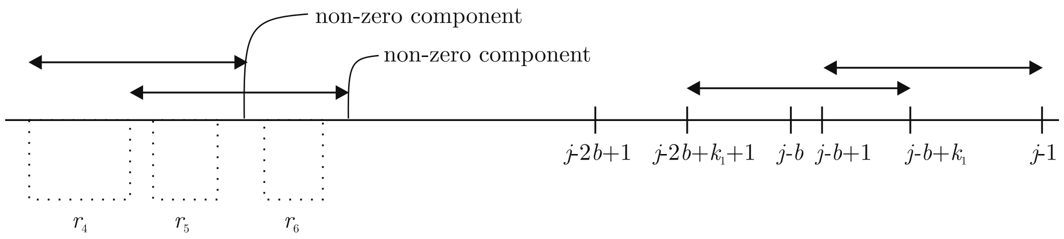
$$\sum_{i=0}^{2} \binom{(j-2b+k_1)-ib+i}{i}(q-1)^i[1+(q-1)]^{i(b-1,w-1)}. \tag{11}$$

Therefore in this case, the total number of choices of coefficients turns out to be

$$\sum_{k_1=1}^{b-1}\{(\text{expr. } (10)) \cdot (\text{expr. } (11))\}. \tag{12}$$

Case 3. *When the h_p are selected from $h_{j-2b+2}, \ldots, h_{j-1}$ such that all the h_p are neither taken from $h_{j-2b+2}, \ldots, h_{j-b}$ nor from $h_{j-b+1}, \ldots, h_{j-1}$, i.e the last*

h_p is among $h_{j-b+1}, \ldots, h_{j-1}$. The h_r and h_ℓ are selected together from $2b-1$ or fewer consecutive columns (but from $b+1$ or more) which are taken from first $j-2b+k_1$ columns, $1 \le k_1 \le b-1$.



In this case, the coefficients δ_p are selected from b consecutive components as w or less non-zero components, which starts from the $(j-2b+1+k_1)$-th component which may obviously continue upto $(j-b+k_1)$-th component. We shall first select $w-1$ or less non-zero components from amongst $(j-2b+1+k_1, \ldots, j-b+k_1-1)$-th positions, the $(j-b+k_1)$-th component is non-zero, together with $w-1$ or less non-zero components amongst $(j-b+1, \ldots, j-1)$-th positions. Such a selection of coefficients δ_p and α_j, following the procedure as in case 2 gives rise to the number of choices of coefficients δ_p and α_j as $L(b, k_1, r_1, r_2, r_3)$ with the constraints stated in (8) and (9). The coefficients β_r and γ_ℓ are selected following the same procedure as in case 2. The last non-zero coefficient of γ_ℓ can be selected in $(q-1)$ ways, thus such a selection of coefficients β_r and γ_ℓ gives us

$$\sum_{k_2=1}^{b-1} \sum_{r_4, r_5, r_6} \binom{b-k_2}{r_4} \binom{k_2-1}{r_5} \binom{b-k_2-1}{r_6} (q-1)^{r_4+r_5+r_6+2} \tag{13}$$

choices,

$$\text{where } 0 \le r_4 \le w-1, \quad 0 \le r_5 \le 2w-2, \quad 0 \le r_6 \le w-1. \tag{14}$$

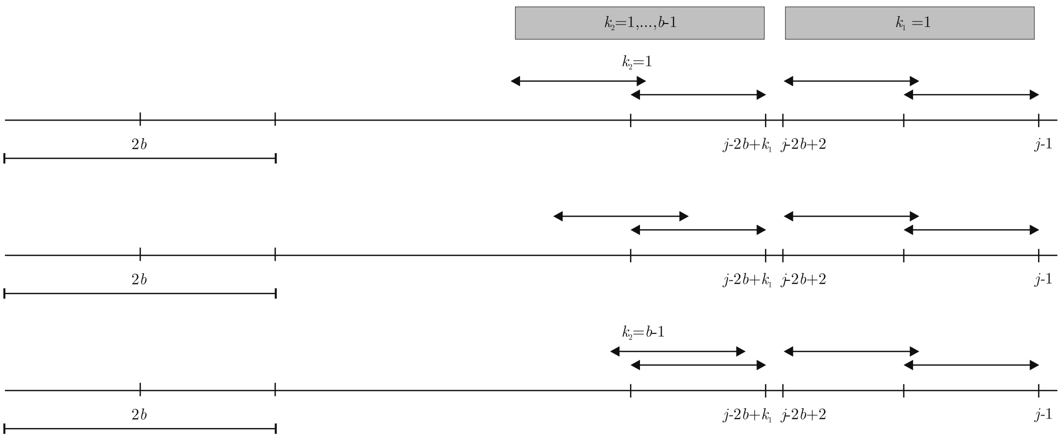
Keeping in view the situations considered in cases 1 and 2, r_4, r_5, r_6 should be such that

$$r_5 + r_6 \geq w - 1, r_4 + r_5 + r_6 \leq 2w - 2. \tag{15}$$

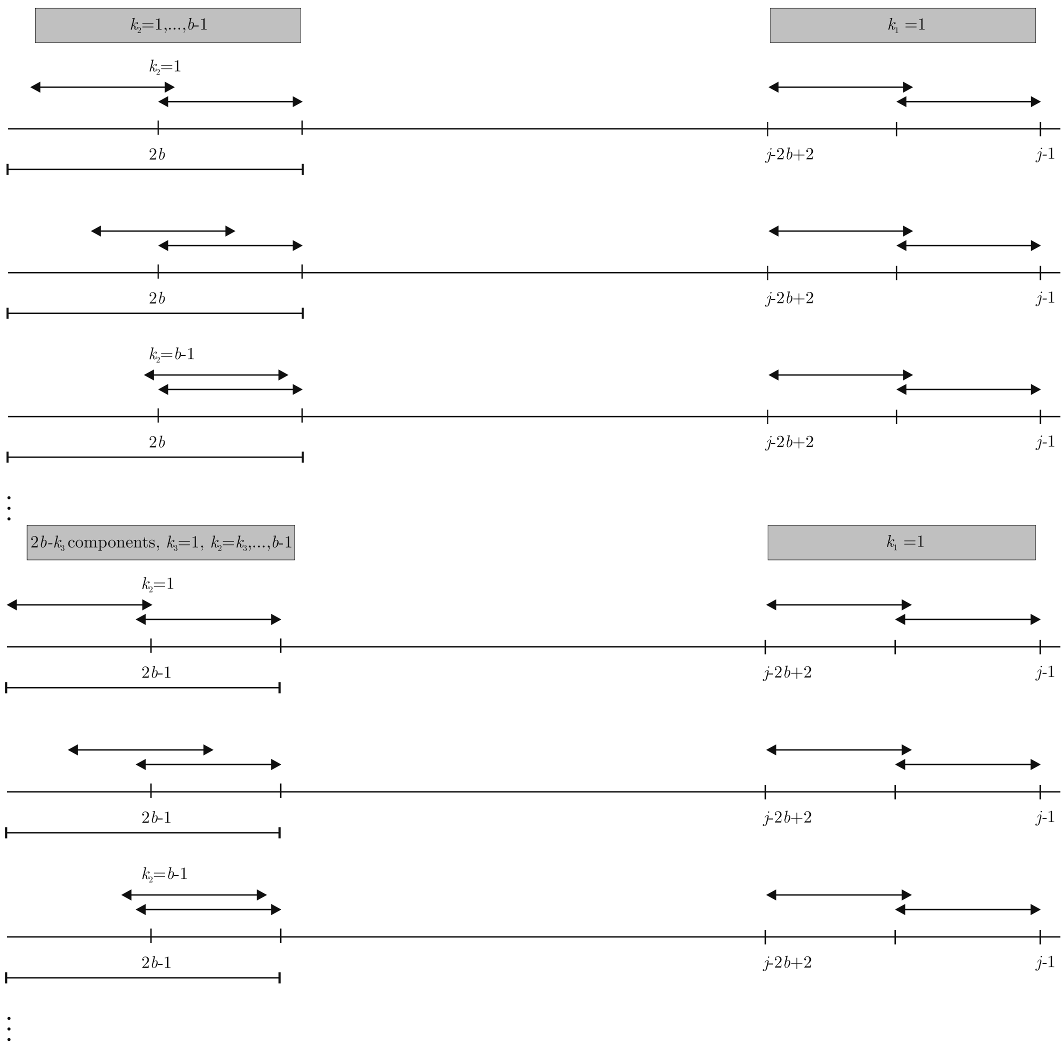
Since the selection of β_r and γ_ℓ is made among the first $(j-2b+k_1)$ components, therefore such a selection of coefficients β_r and γ_ℓ gives us

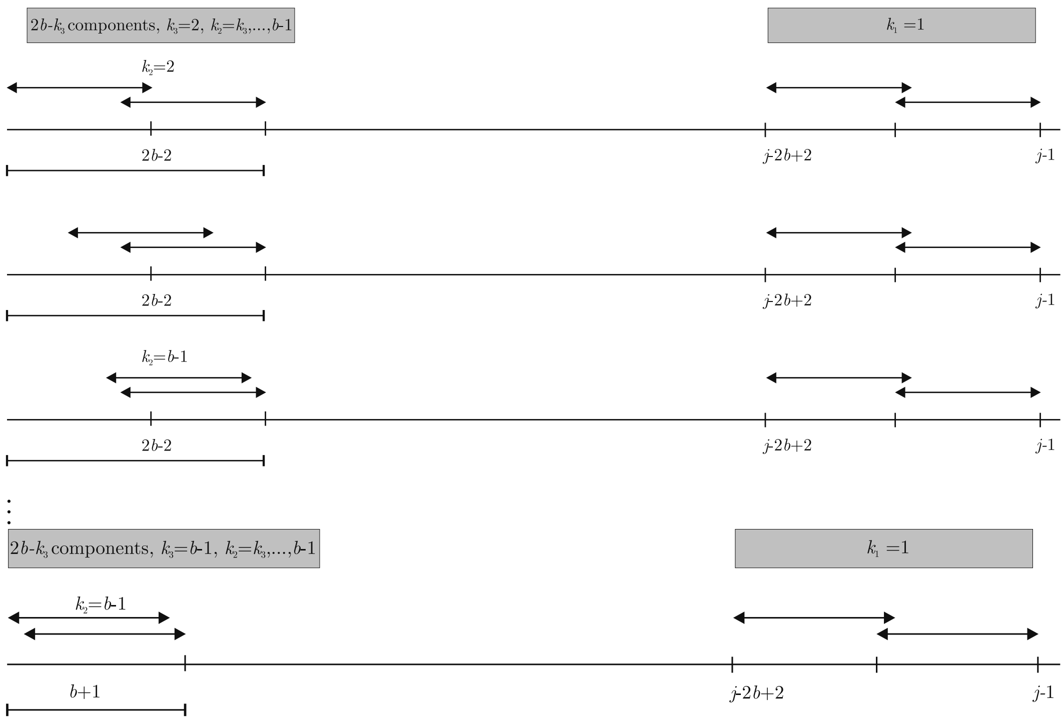
$$\begin{aligned}&\Big\{((j-2b+k_1)-2b+1)\cdot(\text{expr. }(13))\Big\}\\&\quad+\Bigg\{\sum_{\substack{k_2=k_3\\1\leq k_3\leq b-1}}^{b-1}\sum_{r_4,r_5,r_6}\binom{b-k_2}{r_4}\binom{k_2-1}{r_5}\binom{b-k_2-1}{r_6}(q-1)^{r_4+r_5+r_6+2}\Bigg\},\end{aligned} \tag{16}$$

(the last non-zero coefficient during the selection of coefficients β_r, γ_ℓ has the positions $(j-2b+k_1,\ldots,2b)$-th with $1 \leq k_2 \leq b-1$ and further it has $(2b-k_3)$-th position where $k_3 = 1, 2, \ldots, b-1$ with $k_2 = k_3, \ldots, b-1$).



⋮


k_2=1,...,b-1
k_1 =1
k_2=1
2b
j-2b+2
j-1
k_2=b-1
2b-k_3 components, k_3=1, k_2=k_3,...,b-1
2b-1



Similarly, when $k_1 = 2$, the last non-zero coefficient has the positions $(j - 2b+2, \ldots, 2b)$ with $k_2 = 1, \ldots, b-1$ and further it has $(2b - k_3)$-th positions where $k_3 = 1, 2, \ldots, b-1$ with $k_2 = k_3, \ldots, b-1$.

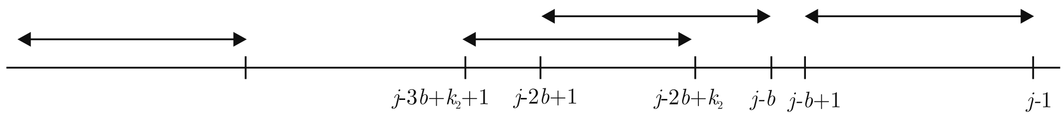
Thus in this case, the total number of choices of coefficients turns out to be

$$\sum_{k_1=1}^{b-1} \big\{(L(b, k_1, r_1, r_2, r_3)) \cdot (\text{expr. } (16))\big\}. \tag{17}$$

The expression $\sum_{r_4, r_5, r_6} \binom{b-k_2}{r_4}\binom{k_2-1}{r_5}\binom{b-k_2-1}{r_6}(q-1)^{r_4+r_5+r_6+2}$ with constraints stated in (14) and (15), is denoted with $L_1(b, k_2, r_4, r_5, r_6)$.

Case 4. When the h_r and h_p are selected together from $2b-1$ or fewer columns (but from $b+1$ or more) from amongst the columns $h_{b+1}, \ldots, h_{j-b}$ i.e. suppose the columns are selected from $h_{j-3b+k_2+1}, \ldots, h_{j-b}$ such that all the h_r

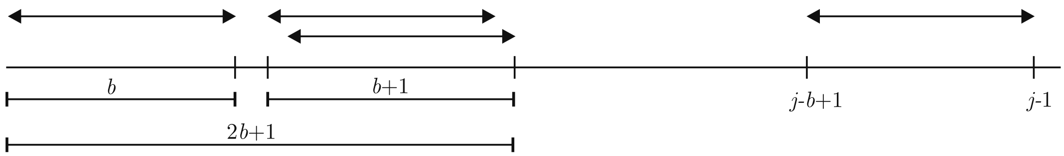
are neither taken from $h_{j-3b+k_2+1}, \ldots, h_{j-2b}$ nor from $h_{j-2b+1}, \ldots, h_{j-b}$, the h_p are selected from $h_{j-2b+1}, \ldots, h_{j-b}$, the h_ℓ are taken from some b consecutive columns amongst the first $j-3b+k_2$ columns, $1 \leq k_2 \leq b-1$.



In this case, the number of ways in which the coefficients α_j can be selected is

$$[1+(q-1)]^{(b-1,w-1)}. \tag{18}$$

The coefficients β_r and δ_p are selected as in case 3 which gives us $L_1(b, k_2, r_4, r_5, r_6)$. The coefficients γ_ℓ as a single low-density burst of length b(fixed) with weight w or less is selected from amongst the first $j-3b+k_2-i+1$ components, $1 \leq k_2 \leq b-1$, i represents the positions $(j-b), \ldots, (2b+1)$.

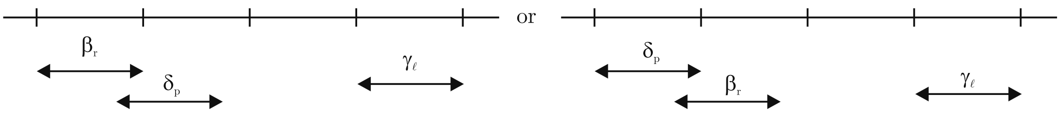


To enumerate the total number of choices of γ_ℓ, β_r and δ_p we prove the following Lemma:

Lemma 1. *$L_1(b, k_2, r_4, r_5, r_6)$ denotes the number of ways for the selection of β_r and δ_p. The total number of choices of γ_ℓ, β_r and δ_p with varying starting position of the first non-zero component when β_r and δ_p are selected together from $2b-1$ or fewer components (but from $b+1$ or more) along with the selection of γ_ℓ which forms a single low-density burst of length b(fixed) with weight*

w or less in the remaining components of the vector of length n_1 (considering $(j-b)$-th position as the first position) is

$$\sum_{i=1}^{n_1-3b+1}\left\{\sum_{k_2=1}^{b-1}\{L_1(b,k_2,r_4,r_5,r_6)\right.$$
$$\left.\times\{((n_1-2b+k_2-i+1)-b+1)(q-1)[1+(q-1)]^{(b-1,w-1)}\}\}\right\}$$
$$+\sum_{\substack{i=n_1-3b+1+k_3\\1\le k_3\le b-1}}\left\{\sum_{k_2=k_3}^{b-1}\{L_1(b,k_2,r_4,r_5,r_6)\right.$$
$$\left.\times\{((n_1-2b+k_2-i+1)-b+1)(q-1)[1+(q-1)]^{(b-1,w-1)}\}\}\right\}. \quad (19)$$



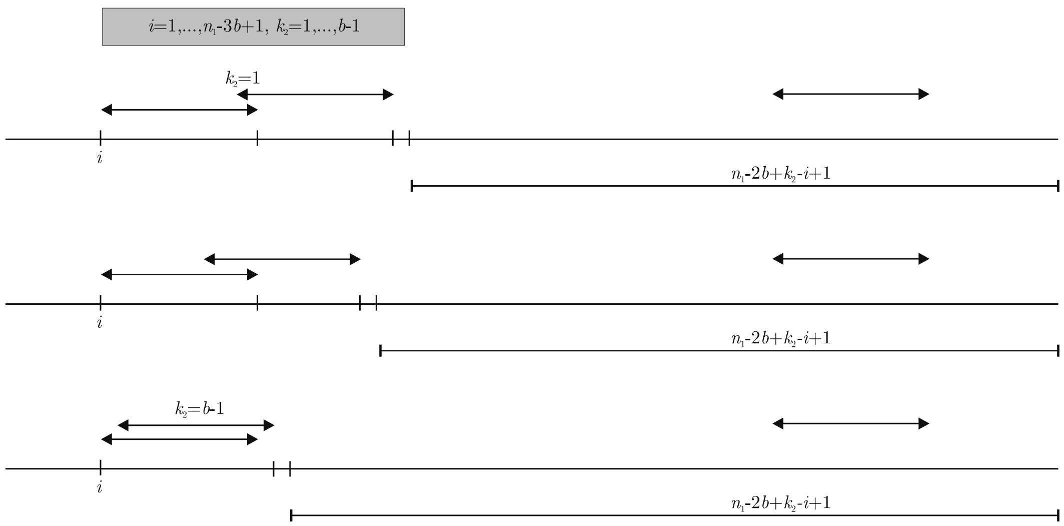
Proof of Lemma 1. Consider a vector of length n_1. When first non-zero component during the selection of β_r and δ_p selected together from $2b-1$ or less components (but from $b+1$ or more) is at the first position then the single low-density burst of length b(fixed) with weight w or less is in the remaining n_1-2b+k_2 components, $1\le k_2\le b-1$. When first non-zero component is at the i-th position then the single low-density burst is in the remaining n_1-2b+k_2-i+1 components, where $1\le i\le n_1-3b+1$. The number for the selection of δ_p and β_r is $L_1(b,k_2,r_4,r_5,r_6)$ as in case 3. The number of single low-density bursts of length b(fixed) with weight w or less in a vector of length (n_1-2b+k_2-i+1) (not including vector of all zeros) is (Dass (1983))

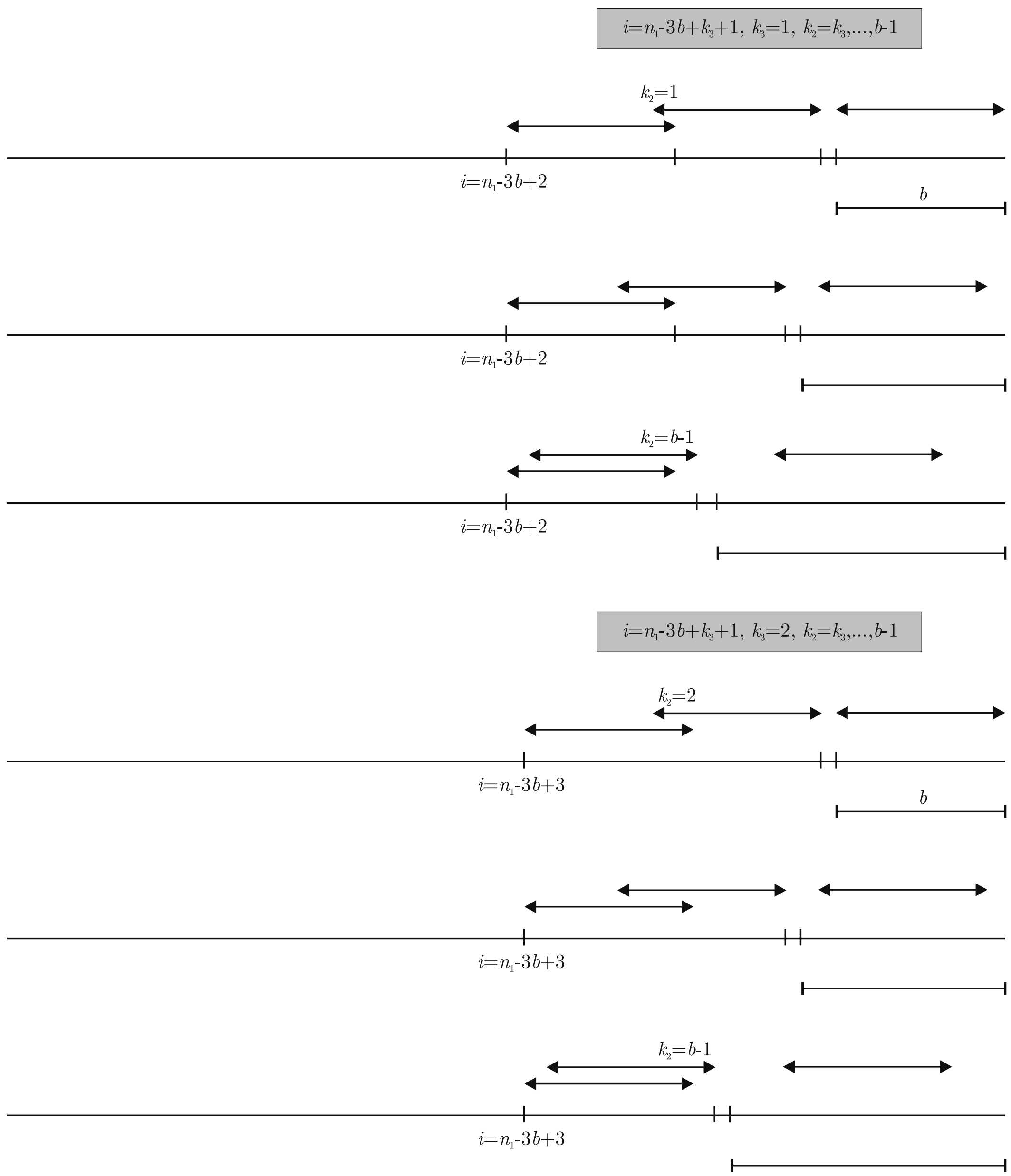
$$((n_1-2b+k_2-i+1)-b+1)(q-1)[1+(q-1)]^{(b-1,w-1)}.$$

Summing on i, the number of such vectors is

$$\sum_{i=1}^{n_1-3b+1}\left\{\sum_{k_2=1}^{b-1}\{L_1(b,k_2,r_4,r_5,r_6)\right.$$
$$\left.\times\{((n_1-2b+k_2-i+1)-b+1)(q-1)[1+(q-1)]^{(b-1,w-1)}\}\}\right\}. \quad (20)$$

Also, when i takes the value n_1-3b+2, k_2 can take values $1,\dots,b-1$, then the single low-density burst of length b(fixed) with weight w or less is in the remaining n_1-2b+k_2-i+1, i.e., when i takes the value n_1-3b+k_3+1, $1\le k_3\le b-1$, with $k_3\le k_2\le b-1$, then single low-density burst of length b(fixed) with weight w or less is in the remaining n_1-2b+k_2-i+1 components.





i=n_1-3b+k_3+1, k_3=1, k_2=k_3,...,b-1
k_2=1
i=n_1-3b+2
b
i=n_1-3b+2
k_2=b-1
i=n_1-3b+2
i=n_1-3b+k_3+1, k_3=2, k_2=k_3,...,b-1
k_2=2
i=n_1-3b+3
b
i=n_1-3b+3
k_2=b-1
i=n_1-3b+3

$\vdots$

Therefore, further summing on i we get

$$\sum_{\substack{i=n_1-3b+k_3+1\\ 1\le k_3\le b-1}} \left\{ \sum_{k_2=k_3}^{b-1} \{L_1(b,k_2,r_4,r_5,r_6) \right.$$

$$\left. \times\{((n_1-2b+k_2-i+1)-b+1)(q-1)[1+(q-1)]^{(b-1,w-1)}\}\} \right\}. \quad (21)$$

Thus, the total number of such vectors is

$$(\text{expr. }(20)) + (\text{expr. }(21)) \quad (22)$$

which completes the proof of Lemma 1.

Now, total number of choices of coefficients β_r, γ_ℓ and δ_p in a vector of length $j-b$ turns out to be (using Lemma 1)

$$\sum_{i=1}^{(j-b)-3b+1} \left\{ \sum_{k_2=1}^{b-1} \{L_1(b,k_2,r_4,r_5,r_6) \right.$$

$$\left. \times\{((j-3b+k_2-i+1)-b+1)(q-1)[1+(q-1)]^{(b-1,w-1)}\}\} \right\}$$

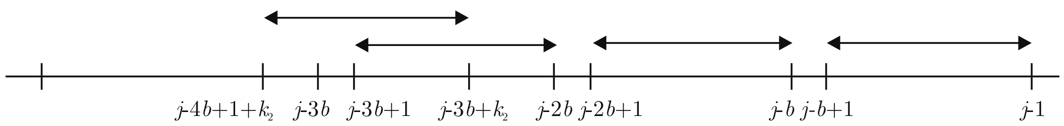
$$+ \sum_{\substack{i=(j-b)-3b+1+k_3\\ 1\le k_3\le b-1}} \left\{ \sum_{k_2=k_3}^{b-1} \{L_1(b,k_2,r_4,r_5,r_6) \right.$$

$$\left. \times \quad \{((j-3b+k_2-i+1)-b+1)(q-1)[1+(q-1)]^{(b-1,w-1)}\}\} \right\}. \quad (23)$$

Therefore, the number of choices of all the coefficients is

$$(\text{expr. }(18)) \cdot (\text{expr. }(23)). \quad (24)$$

Case 5. *When the h_r and h_ℓ are selected together from $2b-1$ or fewer columns (but from $b+1$ or more) from amongst the columns $h_1, \ldots, h_{j-2b}$, i.e, suppose the columns are selected from $h_{j-4b+k_2+1}, \ldots, h_{j-2b}$ such that all h_r are neither taken from $h_{j-4b+k_2+1}, \ldots, h_{j-3b}$ nor from $h_{j-3b+1}, \ldots, h_{j-2b}$, the columns h_ℓ are selected from $h_{j-3b+1}, \ldots, h_{j-2b}$ and the columns h_p are selected from $h_{j-2b+1}, \ldots, h_{j-b}$, $(1 \leq k_2 \leq b-1)$.*



In this case, the number of ways in which the coefficients α_j can be selected is

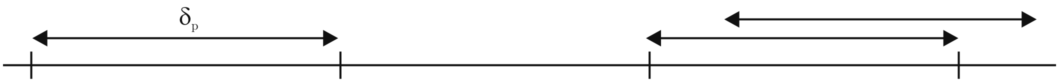
$$[1+(q-1)]^{(b-1,w-1)}. \tag{25}$$

The coefficients β_r and γ_ℓ are selected as in case 3 which gives us $L_1(b, k_2, r_4, r_5, r_6)$. To enumerate the total number of choices of the coefficients δ_p along with β_r and γ_ℓ we prove the following Lemma 2.

Lemma 2. *$L_1(b, k_2, r_4, r_5, r_6)$ denotes the number as in case 3 for the selection of β_r and γ_ℓ. The number of vectors with varying starting position of δ_p which forms single low-density burst of length b(fixed) with weight w or less (consider the position $(j-b)$-th as the first position) along with the selection of β_r and γ_ℓ together from $2b-1$ or fewer components (but from $b+1$ or more) in the*

remaining components of the vector of length n_1, is

$$\sum_{i=1}^{n_1-3b+1} \Bigg\{ \left((q-1)[1+(q-1)]^{(b-1,w-1)}\right)$$
$$\times \Bigg\{ ((n_1-b-i+1)-2b+1) \sum_{k_2=1}^{b-1} L_1(b,k_2,r_4,r_5,r_6)$$
$$+ \sum_{\substack{k_2=k_3\\1\le k_3\le b-1}}^{b-1} L_1(b,k_2,r_4,r_5,r_6)\Bigg\}\Bigg\}$$
$$+ \Bigg\{ \left((q-1)[1+(q-1)]^{(b-1,w-1)}\right) \sum_{\substack{k_2=k_4\\1\le k_4\le b-1}} L_1(b,k_2,r_4,r_5,r_6)\Bigg\}. \quad (26)$$

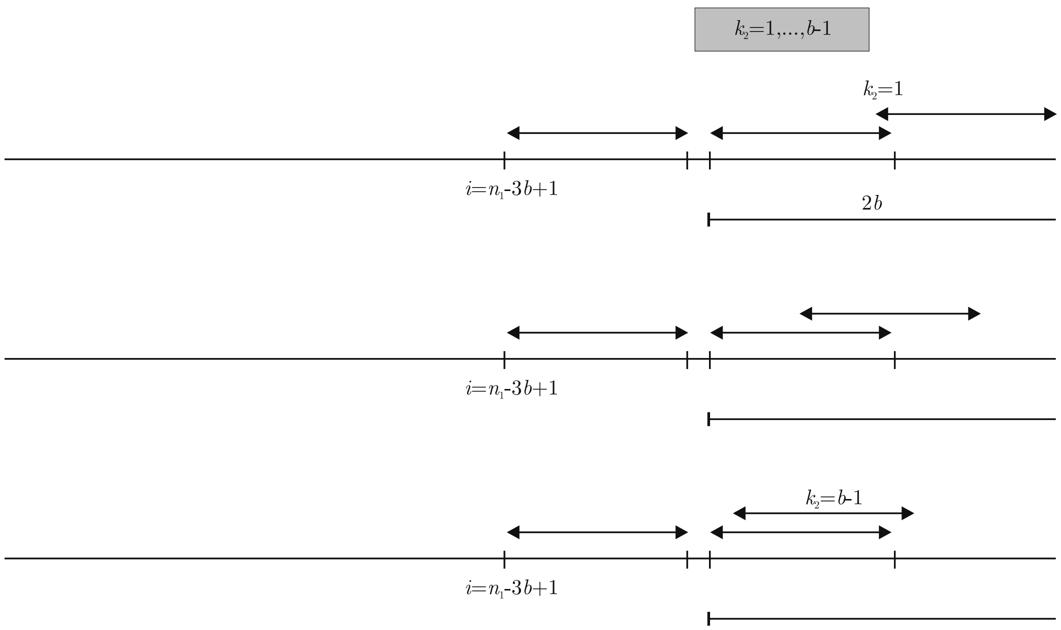


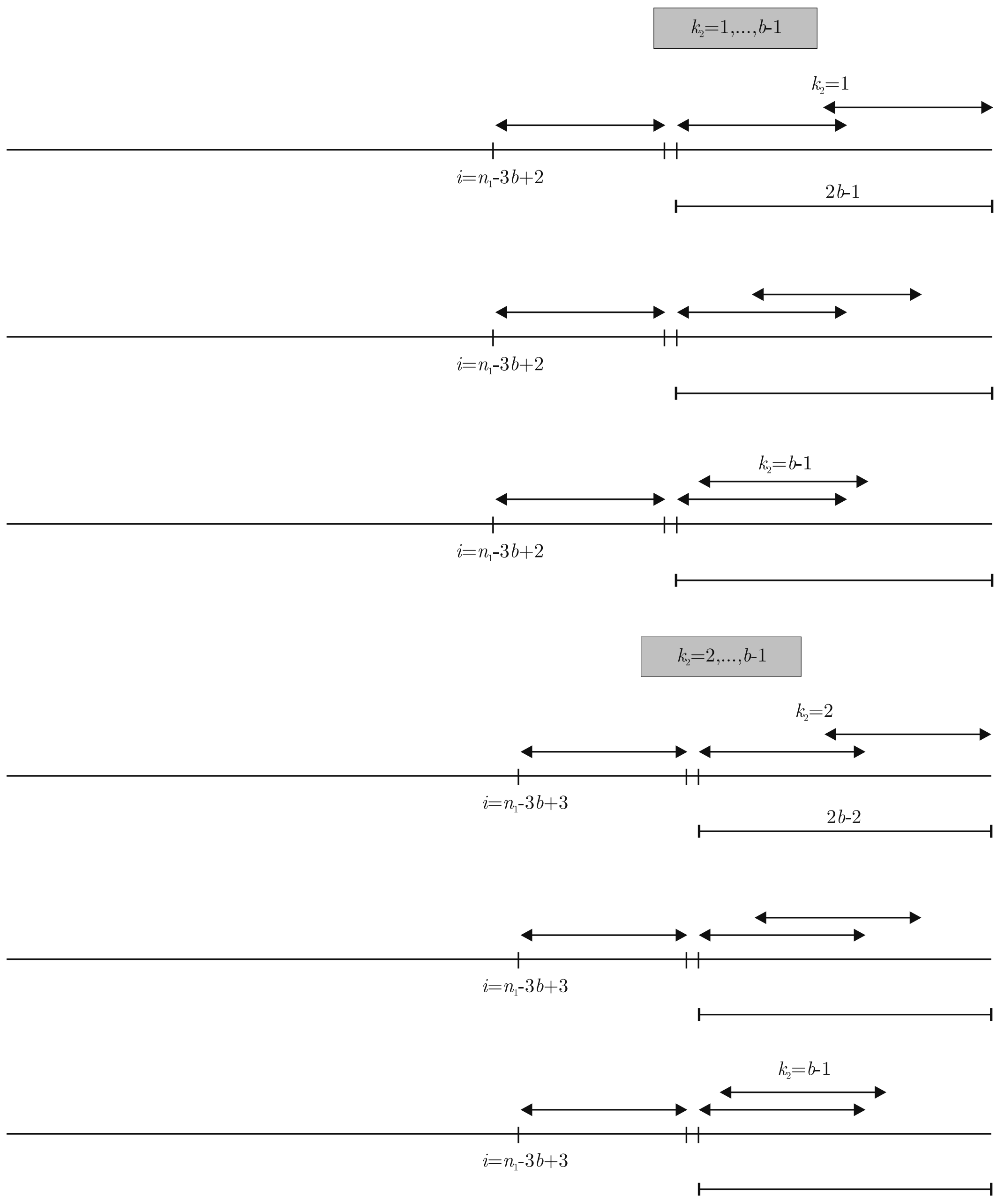
Proof of Lemma 2. Consider a vector of length n_1. When the first non-zero component of δ_p is at the first position then the selection of β_r and γ_ℓ is made (together from $2b-1$ or fewer but from $b+1$ or more components) from the remaining n_1-b components. When the first non-zero component of δ_p is at the i-th position then the selection of β_r and γ_ℓ together is made from the remaining $n_1-b-i+1$ components, $1 \le i \le n_1-3b+1$.

Summing on i, the number of such vectors is

$$\sum_{i=1}^{n_1-3b+1} \Bigg\{ \left((q-1)[1+(q-1)]^{(b-1,w-1)}\right)$$
$$\times \big\{ ((n_1-b-i+1)-2b+1) \sum_{k_2=1}^{b-1} L_1(b,k_2,r_4,r_5,r_6)$$
$$+ \sum_{\substack{k_2=k_3\\1\le k_3\le b-1}}^{b-1} L_1(b,k_2,r_4,r_5,r_6)\big\}\Bigg\}. \quad (27)$$

Further, when i takes the value $n_1 - 3b + 2$, the selection of β_r and γ_ℓ is made from the remaining $n_1 - b - (n_1 - 3b + 2) + 1$ i.e., $2b - 1$ components, k_2 can take the values as $1, \ldots, b-1$, i.e., when i takes the value $n_1 - 3b + k_4 + 1$, $1 \leq k_4 \leq b-1$, selection of β_r and γ_ℓ is made from the remaining $n_1 - b - i + 1$, k_2 can take the values as $k_4, \ldots, b-1$.




k2=1,...,b-1
k2=1
i=n1-3b+2
2b-1
i=n1-3b+2
k2=b-1
i=n1-3b+2
k2=2,...,b-1
k2=2
i=n1-3b+3
2b-2
i=n1-3b+3
k2=b-1
i=n1-3b+3

⋮

Therefore, further summing on i we get

$$\sum_{\substack{i=n_1-3b+1+k_4\\1\le k_4\le b-1}} \left\{\left((q-1)[1+(q-1)]^{(b-1,w-1)}\right)\sum_{k_2=k_4}^{b-1} L_1(b,k_2,r_4,r_5,r_6)\right\}$$

i.e.

$$\left\{\left((q-1)[1+(q-1)]^{(b-1,w-1)}\right)\sum_{\substack{k_2=k_4\\1\le k_4\le b-1}}^{b-1} L_1(b,k_2,r_4,r_5,r_6)\right\}. \tag{28}$$

Thus, total number of such vectors is

$$(\text{expr. }(27)) + (\text{expr. }(28)) \tag{29}$$

which completes the proof of the Lemma 2.

Now, the number of choices of coefficients β_r, γ_ℓ and δ_p in a vector of length

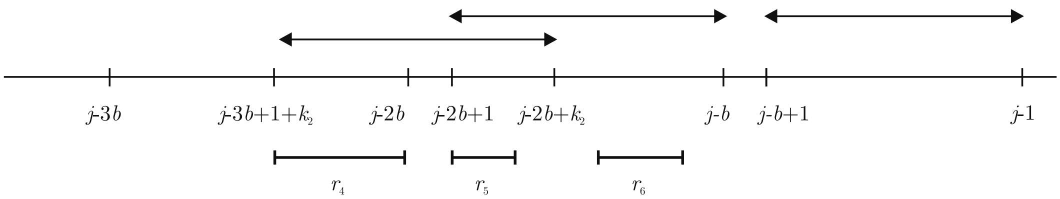
$j-b$ is obtained by replacing n_1 by $j-b$ in expression (29) giving rise to

$$\sum_{i=1}^{(j-b)-3b+1}\Bigg\{((q-1)[1+(q-1)]^{(b-1,w-1)})$$
$$\times\Bigg\{((j-2b-i+1)-2b+1)\sum_{k_2=1}^{b-1}L_1(b,k_2,r_4,r_5,r_6)$$
$$+\sum_{\substack{k_2=k_3\\1\le k_3\le b-1}}^{b-1}L_1(b,k_2,r_4,r_5,r_6)\Bigg\}\Bigg\}$$
$$+\Bigg\{((q-1)[1+(q-1)]^{(b-1,w-1)})$$
$$\times\sum_{\substack{k_2=k_4\\1\le k_4\le b-1}}^{b-1}L_1(b,k_2,r_4,r_5,r_6)\Bigg\}. \tag{30}$$

Therefore, the total number of choices of all the coefficients is

$$(\text{expr. }(25))\cdot(\text{expr. }(30)). \tag{31}$$

Case 6. *When the h_r and h_ℓ are selected together from $2b-1$ or fewer columns (but from $b+1$ or more) from amongst the columns $h_1,\ldots,h_{j-b}$ i.e. suppose the h_r and h_ℓ are selected from $h_{j-3b+1+k_2},\ldots,h_{j-b}$, the h_ℓ are selected from $h_{j-2b+1},\ldots,h_{j-b}$ and the h_r are selected from $h_{j-3b+1+k_2},\ldots,h_{j-2b+k_2}$ such that all the h_r are neither from $h_{j-3b+1+k_2},\ldots,h_{j-2b}$ nor from $h_{j-2b+1},\ldots,h_{j-b}$, $1\le k_2\le b-1$.*



In this case, the number of ways in which the coefficients α_j can be selected is

$$[1+(q-1)]^{(b-1,w-1)}. \tag{32}$$

The number of ways in which the coefficients β_r and γ_ℓ are selected is $L_1(b,k_2,r_4,r_5,r_6)$ (refer case 3). Further, β_r and γ_ℓ are to be selected such that the last non-zero coefficient of γ_ℓ can take position $j-b, j-b-1, \dots, b+1$ in a vector of length $j-b$.

To enumerate total number of choices of the coefficients β_r and γ_ℓ we prove the following Lemma 3 (consider the position $(j-b)$-th as the first position).

Lemma 3. *$L_1(b,k_2,r_4,r_5,r_6)$ represents the number as in case 3 for the selection of γ_ℓ and β_r. The number of such vectors with the varying starting position of γ_ℓ in a vector of length n_1 is*

$$(n_1-2b+1)\left(\sum_{k_2=1}^{b-1} L_1(b,k_2,r_4,r_5,r_6)\right) + \sum_{\substack{k_2=k_4\\ 1\le k_4\le b-1}}^{b-1} L_1(b,k_2,r_4,r_5,r_6)\,. \tag{33}$$

Proof of Lemma 3. Consider a vector of length n_1. The first non-zero component can take the position $i=1,\dots,n_1-b$. Obviously, for $i=1,\dots,n_1-2b+1$, k_2 can take values as $1\le k_2\le b-1$ and for $i=n_1-2b+2,\dots,n_1-b$ i.e. for $i=n_1-2b+1+k_4$, $1\le k_4\le b-1$, k_2 can take values as $k_4\le k_2\le b-1$. Thus total number of such vectors is

$$(n_1-2b+1)\left(\sum_{k_2=1}^{b-1} L_1(b,k_2,r_4,r_5,r_6)\right) + \sum_{\substack{k_2=k_4\\ 1\le k_4\le b-1}}^{b-1} L_1(b,k_2,r_4,r_5,r_6)$$

which completes the proof of Lemma 3.

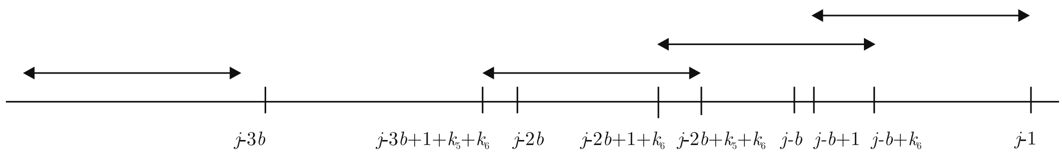
Now the number of choices of the coefficients β_r and γ_ℓ in a vector of length $j-b$ can be obtained by replacing n_1 by $j-b$ in expression (33) which gives

$$(j-3b+1)\sum_{k_2=1}^{b-1} L_1(b,k_2,r_4,r_5,r_6) + \sum_{\substack{k_2=k_4 \\ 1\le k_4\le b-1}}^{b-1} L_1(b,k_2,r_4,r_5,r_6)\,. \tag{34}$$

Therefore, total number of choices of all the coefficients is

$$(\text{expr. }(32))\cdot(\text{expr. }(34)). \tag{35}$$

Case 7. *When the h_r, h_p and h_j are selected together from $3b-3$ or fewer columns (but from $2b-1$ or more) i.e. the h_p are selected from b consecutive columns amongst $h_{j-2b+2},\ldots,h_{j-1}$ with the last h_p among $h_{j-b+1},\ldots h_{j-1}$ and the h_r are selected from b consecutive columns amongst $h_{j-3b+3},\ldots,h_{j-b}$ with the last h_r among $h_{j-2b+1+k_6,\ldots,h_{j-b}}$ (depending on the selection of h_p's), $1\le k_6\le b-1$, together with the h_ℓ being selected from some b consecutive columns from the first $j-3b+1+k_7$ columns, $1\le k_7\le b-1$ (depending upon the b consecutive columns from which h_r's are selected).*



In this case, the coefficients δ_p are selected from b consecutive components as w or less non-zero components, which starts from $(j-2b+1+k_6)$-th component which may obviously continue upto $(j-b+k_6)$-th component. The coefficients β_r are selected from b consecutive components as w or less non-zero components, which starts from $(j-3b+1+k_5+k_6)$-th component which may

continue obviously upto $(j - 2b + k_5 + k_6)$-th component, $1 \leq k_6 \leq b - 1$, $1 \leq k_5 \leq b - k_6$. Suppose $k_5 + k_6 - 1 = k_7$.

Our main objective is to select $w - 1$ or less non-zero components amongst $(j - 3b + 1 + k_5 + k_6, \ldots, j - 2b + k_5 + k_6 - 1)$-th positions, $(j - 2b + k_5 + k_6)$-th component is non-zero, $w - 1$ or less non-zero components amongst $(j - 2b + k_6 + 1, \ldots, j - b + k_6 - 1)$-th positions, $(j - b + k_6)$-th component is non-zero and $w - 1$ or less non-zero components amongst $(j - b + 1, \ldots, j - 1)$-th positions. Also we have to select the coefficients γ_ℓ which appear as w or less non-zero components from a set of b consecutive components among the first $j - 3b + k_5 + k_6$ components where $1 \leq k_6 \leq b - 1$, $1 \leq k_5 \leq b - k_6$.

In order to do so, let us choose

r_7 components from the $(j - 3b + 1 + k_5 + k_6, \ldots, j - 2b + k_6)$-th positions,

r_8 components from the $(j - 2b + k_6 + 1, \ldots, j - 2b + k_5 + k_6 - 1)$-th positions,

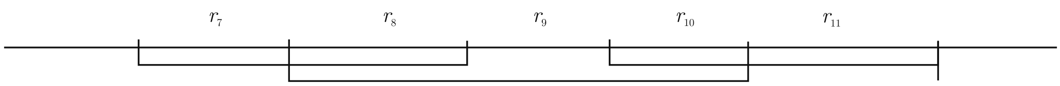
r_9 components from the $(j - 2b + k_5 + k_6 + 1, \ldots, j - b)$-th positions,

r_{10} components from the $(j - b + 1, \ldots, j - b + k_6 - 1)$-th positions,

r_{11} components from the $(j - b + k_6 + 1, \ldots, j - 1)$-th positions,

where

$$0 \leq r_7 \leq w - 1,\ 0 \leq r_8 \leq 2w - 2,\ 0 \leq r_9 \leq w - 1,\ 0 \leq r_{10} \leq 2w - 2,$$
$$0 \leq r_{11} \leq w - 1. \tag{36}$$



Keeping in view the situations considered in cases 1, 2 and 4, $r_7, r_8, r_9, r_{10}, r_{11}$

should be such that

$$r_{10}+r_{11}\geq w-1, r_8+r_9+r_{10}+r_{11}\geq 2w-2,$$
$$r_7+r_8+r_9+r_{10}+r_{11}\leq 3w-3. \tag{37}$$

Such a selection of coefficients give us

$$\sum_{\substack{r_7,r_8,r_9,\\ r_{10},r_{11}}}\binom{b-k_5}{r_7}\binom{k_5-1}{r_8}\binom{b-k_5-k_6}{r_9}\binom{k_6-1}{r_{10}} \times\binom{b-k_6-1}{r_{11}}(q-1)^{r_7+r_8+r_9+r_{10}+r_{11}}$$

possible linear combinations where $r_7, r_8, r_9, r_{10}, r_{11}$ each satisfy the constraints stated in (36) and (37). The $(j-2b+k_5+k_6)$-th and $(j-b+k_6)$-th components can be selected in $(q-1)$ ways each, therefore selection of coefficients give us

$$\sum_{\substack{r_7,r_8,r_9,\\ r_{10},r_{11}}}\binom{b-k_5}{r_7}\binom{k_5-1}{r_8}\binom{b-k_5-k_6}{r_9}\binom{k_6-1}{r_{10}} \times\binom{b-k_6-1}{r_{11}}(q-1)^{r_7+r_8+r_9+r_{10}+r_{11}+2}. \tag{38}$$

Now to select the coefficients γ_ℓ it is equivalent to enumerate the single low-density burst of length b(fixed) with weight w or less in a vector of length $j-3b+k_5+k_6$, which gives us (Dass (1983))

$$1+((j-3b+k_5+k_6)-b+1)(q-1)[1+(q-1)]^{(b-1,w-1)}. \tag{39}$$

Therefore, in this case total number of choices of coefficients turns out to be

$$\sum_{\substack{k_5,k_6\\ 1\leq k_6\leq b-1\\ 1\leq k_5\leq b-k_6}} (\text{expr. (38)})\cdot(\text{expr. (39)}). \tag{40}$$

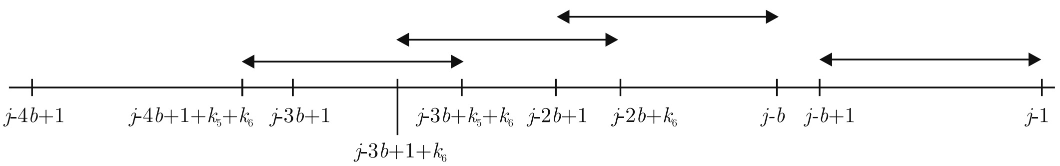
The expression

$$\sum_{\substack{r_7,r_8,r_9,\\ r_{10},r_{11}}} \binom{b-k_5}{r_7}\binom{k_5-1}{r_8}\binom{b-k_5-k_6}{r_9}\binom{k_6-1}{r_{10}} \times \binom{b-k_6-1}{r_{11}}(q-1)^{r_7+r_8+r_9+r_{10}+r_{11}+2}$$

with constraints stated in (36) and (37) is denoted as

$$L_2(b, k_5, k_6, r_7, r_8, r_9, r_{10}, r_{11}).$$

Case 8. *When the h_ℓ, h_r, h_p are selected together from $3b-2$ or fewer columns (but from $2b$ or more) from amongst the first $j-b$ columns, i.e., suppose the h_p are selected from b consecutive columns amongst $h_{j-2b+1}, \ldots, h_{j-b}$, the h_r are selected from b consecutive columns amongst $h_{j-3b+2}, \ldots, h_{j-b-1}$ with the last h_r among $h_{j-2b+1}, \ldots, h_{j-b-1}$ and the h_ℓ are selected from b consecutive columns amongst $h_{j-4b+3}, \ldots, h_{j-2b}$ with last column h_ℓ among $h_{j-3b+1+k_6}, \ldots, h_{j-2b}$ (depending on the selection of h_r's), $1 \le k_6 \le b-1$.*



In this case, the number of ways in which the coefficients α_j can be selected is

$$[1+(q-1)]^{(b-1,w-1)}. \tag{41}$$

The coefficients δ_p are selected as w or less non-zero components from b consecutive components starting from $(j-2b+1)$-th component which may obviously

continue upto $(j-b)$-th component. The coefficients β_r are selected from the components starting from $(j-3b+1+k_6)$-th component which may obviously continue upto $(j-2b+k_6)$-th component $(1 \leq k_6 \leq b-1)$ as w or less non-zero components from b consecutive components. The coefficients γ_ℓ are selected as w or less non-zero components from $(j-4b+1+k_5+k_6)$-th component which may obviously continue upto $(j-3b+k_5+k_6)$-th component, $1 \leq k_6 \leq b-1, 1 \leq k_5 \leq b-k_6$.

Our main objective is to select $w-1$ or less non-zero components amongst $(j-4b+1+k_5+k_6, \ldots, j-3b+k_5+k_6-1)$-th positions, the $(j-3b+k_5+k_6)$-th component is non-zero, $w-1$ or less non-zero components amongst $(j-3b+1+k_6, \ldots, j-2b+k_6-1)$-th positions, the $(j-2b+k_6)$-th component is non-zero, and $w-1$ or less non-zero components amongst $(j-2b+1, \ldots, j-b-1)$-th positions, the $(j-b)$-th component is non-zero. Following the procedure for the selection of coefficients γ_ℓ, β_r and δ_p as in case 7, with the selection of $(j-b)$-th component in $(q-1)$ ways, we get the expression as

$$\sum_{\substack{r_7,r_8,r_9,\\ r_{10},r_{11}}} \binom{b-k_5}{r_7}\binom{k_5-1}{r_8}\binom{b-k_5-k_6}{r_9}\binom{k_6-1}{r_{10}} \times \binom{b-k_6-1}{r_{11}}(q-1)^{r_7+r_8+r_9+r_{10}+r_{11}+3} \tag{42}$$

where

r_7 components are chosen from the $(j-4b+1+k_5+k_6, \ldots, j-3b+k_6)$-th positions,

r_8 components are chosen from the $(j-3b+k_6+1, \ldots, j-3b+k_5+k_6-1)$-th positions,

r_9 components are chosen from the $(j-3b+k_5+k_6+1, \ldots, j-2b)$-th positions,

r_{10} components are chosen from the $(j-2b+1, \ldots, j-2b+k_6-1)$-th positions,

r_{11} components are chosen from the $(j-2b+k_6+1, \ldots, j-b-1)$-th positions,

where

$$\begin{aligned}&0 \leq r_7 \leq w-1, 0 \leq r_8 \leq 2w-2, 0 \leq r_9\\&\leq w-1, 0 \leq r_{10} \leq 2w-2, 0 \leq r_{11} \leq w-1.\end{aligned} \tag{43}$$

Keeping in view the situations considered in cases 1, 4 and 5, r_7, r_8, r_9, r_{10}, r_{11} should be such that

$$\begin{aligned}&r_{10}+r_{11} \geq w-1, \quad r_8+r_9+r_{10}+r_{11} \geq 2w-2,\\&r_7+r_8+r_9+r_{10}+r_{11} \leq 3w-3.\end{aligned} \tag{44}$$

The (expr. (42)) with constraints stated in (43) and (44) is denoted by $L_3(b, k_5, k_6, r_7, r_8, r_9, r_{10}, r_{11})$.

The non-zero component at $(j-b)$-th position during the selection of coefficients, can take positions $(j-b), (j-b-1), \ldots, 2b$.

Total number of ways in which β_r, γ_ℓ and δ_p are selected is

$$\begin{aligned}&((j-b)-3b+3) \sum_{\substack{k_5,k_6\\1\leq k_6\leq b-1\\1\leq k_5\leq b-k_6}} L_3(b, k_5, k_6, r_7, r_8, r_9, r_{10}, r_{11})\\&\quad+\sum_{\substack{i_1,k_5,k_6\\1\leq i_1\leq b-2\\i_1+1\leq k_6\leq b-1\\1\leq k_5\leq b-k_6}} L_3(b, k_5, k_6, r_7, r_8, r_9, r_{10}, r_{11})\\&\quad+\sum_{\substack{i_1,k_5,k_6\\1\leq i_1\leq b-2\\1\leq k_6\leq b-1\\i_1+1\leq k_5\leq b-k_6}} L_3(b, k_5, k_6, r_7, r_8, r_9, r_{10}, r_{11})\end{aligned} \tag{45}$$

with last non-zero component during the selection of coefficients taking positions

(i) $j-b, \ldots, 3b-2$, values of k_5, k_6 varies as $1 \leq k_6 \leq b-1$, $1 \leq k_5 \leq b-k_6$, further

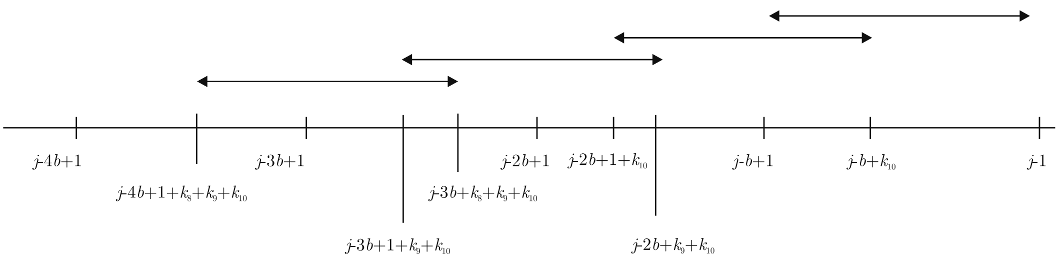
(ii) (a) $(3b-2-i_1), 1 \leq i_1 \leq b-2$ positions, values of k_5, k_6 varies as $i_1+1 \leq k_6 \leq b-1$, $1 \leq k_5 \leq b-k_6$.

(b) $(3b-2-i_1), 1 \leq i_1 \leq b-2$ positions, values of k_5, k_6 varies as $1 \leq k_6 \leq b-1$, $i_1+1 \leq k_5 \leq b-k_6$.

Therefore, in this case total number of choices of coefficients turns out to be

$$(\text{expr. } (41)) \cdot (\text{expr. } (45)). \tag{46}$$

Case 9. *When the h_ℓ, h_r, h_p and h_j are selected together from $4b-4$ or fewer columns (but from $2b$ or more), the h_p are selected from b consecutive columns amongst $h_{j-2b+2}, \ldots, h_{j-1}$ with the last h_p among $h_{j-b+1}, \ldots, h_{j-1}$, the h_r are selected from b consecutive columns amongst $h_{j-3b+3}, \ldots, h_{j-b}$ with the last h_r among $h_{j-2b+1+k_{10}}, \ldots, h_{j-b}$ (depending on the selection of h_p's), $1 \leq k_{10} \leq b-1$, the h_ℓ are selected from b consecutive columns from amongst $h_{j-4b+4}, \ldots, h_{j-2b+k_{10}}$, with last column h_ℓ among $h_{j-3b+1+k_9+k_{10}}, \ldots, h_{j-2b+k_{10}}, 1 \leq k_{10} \leq b-1, 1 \leq k_9 \leq b-k_{10}$, i.e., among the starting of selection of b consecutive columns for h_r upto one column before the starting of selection of h_p.*



In this case, the coefficients δ_p are selected from b consecutive components as w or less non-zero components which start from $(j-2b+1+k_{10})$-th component and may obviously continue upto $(j-b+k_{10})$-th component,

$1 \leq k_{10} \leq b-1$. The coefficients β_r are selected from b consecutive components as w or less non-zero components which start from $(j-3b+1+k_9+k_{10})$-th component and may obviously continue upto $(j-2b+k_9+k_{10})$-th component, $1 \leq k_{10} \leq b-1, 1 \leq k_9 \leq b-k_{10}$. The coefficients γ_ℓ are selected from b consecutive components as w or less non-zero components which start from $(j-4b+1+k_8+k_9+k_{10})$-th component which may obviously continue upto $(j-3b+k_8+k_9+k_{10})$-th component, $1 \leq k_{10} \leq b-1$, $1 \leq k_9 \leq b-k_{10}$, $1 \leq k_8 \leq b-k_9$.

Our main objective is to select $w-1$ or less non-zero components amongst $(j-4b+1+k_8+k_9+k_{10}, \ldots, j-3b+k_8+k_9+k_{10}-1)$-th positions, the $(j-3b+k_8+k_9+k_{10})$-th component is non-zero, $w-1$ or less non-zero components among $(j-3b+1+k_9+k_{10}, \ldots, j-2b+k_9+k_{10}-1)$-th positions, the $(j-2b+k_9+k_{10})$-th component is non-zero, $w-1$ or less non-zero components among $(j-2b+1+k_{10}, \ldots, j-b+k_{10}-1)$-th positions, the $(j-b+k_{10})$-th component is non-zero and $w-1$ or less non-zero components amongst $(j-b+1, \ldots, j-1)$-th positions.

In order to do so, let us choose

r_{12} components from the $(j-4b+1+k_8+k_9+k_{10}, \ldots, j-3b+k_9+k_{10})$-th positions,

r_{13} components from the $(j-3b+k_9+k_{10}+1, \ldots, j-3b+k_8+k_9+k_{10}-1)$-th positions,

r_{14} components from the $(j-3b+k_8+k_9+k_{10}+1, \ldots, j-2b+k_{10})$-th positions,

r_{15} components from the $(j-2b+k_{10}+1, \ldots, j-2b+k_9+k_{10}-1)$-th positions,

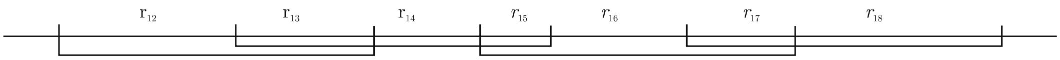
r_{16} components from the $(j-2b+k_9+k_{10}+1, \ldots, j-b)$-th positions,

r_{17} components from the $(j-b+1, \ldots, j-b+k_{10}-1)$-th positions,

r_{18} components from the $(j-b+k_{10}+1,\ldots,j-1)$-th positions,

where

$$0 \le r_{12} \le w-1, 0 \le r_{13} \le 2w-2, 0 \le r_{14} \le w-1,$$
$$0 \le r_{15} \le 2w-2, 0 \le r_{16} \le w-1, 0 \le r_{17} \le 2w-2, 0 \le r_{18} \le w-1. \tag{47}$$



Keeping in view the situations considered in cases 3, 7 and 8, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18} should be such that

$$\begin{aligned} & r_{17}+r_{18} \ge w-1, r_{15}+r_{16}+r_{17}+r_{18} \ge 2w-2, \\ & r_{13}+r_{14}+r_{15}+r_{16}+r_{17}+r_{18} \ge 3w-3, \\ & r_{12}+r_{13}+r_{14}+r_{15}+r_{16}+r_{17}+r_{18} \le 4w-4. \end{aligned} \tag{48}$$

Such a selection of coefficients give us

$$\sum_{r_{12},r_{13},\ldots,r_{18}} \binom{b-k_8}{r_{12}}\binom{k_8-1}{r_{13}}\binom{b-k_8-k_9}{r_{14}}\binom{k_9-1}{r_{15}}\binom{b-k_9-k_{10}}{r_{16}} \times \binom{k_{10}-1}{r_{17}}\binom{b-k_{10}-1}{r_{18}}(q-1)^{r_{12}+r_{13}+r_{14}+r_{!5}+r_{16}+r_{17}+r_{18}} \tag{49}$$

possible linear combinations where $r_{12},\ldots,r_{18}$ each satisfy the constraints stated in (47) and (48). The $(j-b+k_{10})$-th, $(j-2b+k_9+k_{10})$-th and $(j-3b+k_8+k_9+k_{10})$-th components can be selected in $(q-1)$ ways each, therefore selection of coefficients give us

$$\sum_{\substack{k_8,k_9,k_{10} \\ 1\le k_{10}\le b-1 \\ 1\le k_9\le b-k_{10} \\ 1\le k_8\le b-k_9}} \sum_{r_{12},\ldots,r_{18}} \binom{b-k_8}{r_{12}}\binom{k_8-1}{r_{13}}\binom{b-k_8-k_9}{r_{14}}\binom{k_9-1}{r_{15}}\binom{b-k_9-k_{10}}{r_{16}} \times \binom{k_{10}-1}{r_{17}}\binom{b-k_{10}-1}{r_{18}}(q-1)^{r_{12}+r_{13}+r_{14}+r_{!5}+r_{16}+r_{17}+r_{18}+3}. \tag{50}$$

The expression

$$\sum_{r_{12},\dots,r_{18}} \binom{b-k_8}{r_{12}}\binom{k_8-1}{r_{13}}\binom{b-k_8-k_9}{r_{14}}\binom{k_9-1}{r_{15}}\binom{b-k_9-k_{10}}{r_{16}}$$
$$\times\binom{k_{10}-1}{r_{17}}\binom{b-k_{10}-1}{r_{18}}(q-1)^{r_{12}+r_{13}+r_{14}+r_{!5}+r_{16}+r_{17}+r_{18}+3}$$

with constraints stated in (47) and (48) is denoted by
$L_4(b,k_8,k_9,k_{10},r_{12},r_{13},r_{14},r_{15},r_{16},r_{17},r_{18})$.

Thus, the total number of possible combinations that h_j cannot be equal to, is

$$\begin{aligned}&(\text{expr. }(7)) + (\text{expr. }(12)) + (\text{expr. }(17))\\ &\quad +(\text{expr. }(24)) + (\text{expr. }(31)) + (\text{expr. }(35))\\ &\quad +(\text{expr. }(40)) + (\text{expr. }(46)) + (\text{expr. }(50))\,. \end{aligned} \tag{51}$$

At worst, all these linear combinations may yield a distinct sum. Therefore a column h_j can be added to H' provided that

$$q^{n-k} > (\text{expr. }(51)). \tag{52}$$

Now reverse the columns of H' to obtain the requisite parity-check matrix $H = [H_1, H_2 \dots H_n], (h_i \to H_{n-i+1})$. Thus, to achieve code of length n, replace j by n which gives the result.

Remark 1. The result just obtained holds for $w \le b$. If we take $w = b$, the weight consideration over the burst becomes redundant. The situations giving rise to the expressions except (expr. (7)) does not arise. The bound

then reduces to

$$\begin{aligned} q^{n-k} &> [1+(q-1)]^{(b-1,b-1)} \\ &\quad \times \left\{ \sum_{i=0}^{3} \binom{n-(i+1)b+i}{i} (q-1)^i [1+(q-1)]^{i(b-1,b-1)} \right\} \end{aligned}$$

i.e.

$$q^{n-k} > q^{b-1} \left\{ \sum_{i=0}^{3} \binom{n-(i+1)b+i}{i} (q-1)^i q^{i(b-1)} \right\}$$

which coincides with a result due to Dass, Garg and Zannetti (2008b), when bursts considered are 2-repeated bursts of length b(fixed).

We conclude this section with an example.

Example 1. Consider the following 12×16 matrix over GF(2)

$$H = \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1 \\ 1&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0 \\ 1&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0 \\ 1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0 \\ 1&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0 \\ 1&1&0&0&0&0&0&0&0&1&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0 \\ 0&1&1&0&0&0&0&0&0&0&0&1&0&0&1&0 \\ 1&1&0&0&0&0&0&0&0&0&0&0&1&0&0&1 \end{bmatrix}.$$

This matrix has been constructed by the synthesis procedure outlined in the proof of Theorem 1 by taking $b = 3$ and $w = 2$. Considered as a parity-check matrix, this matrix gives rise to a $(16, 4)$ binary code. It can be seen from Table 1 that the syndromes of bursts of length 3(fixed) with weight 2 or less are distinct, showing thereby that the code that is the null space of the matrix

given in the example corrects all bursts of length 3(fixed) with weight 2 or less. It should be noted that this code does not correct all bursts of length 3(fixed) with weight 3, e.g.,

(0000000011100111) as its syndrome is the same as that of (1100000000001000),
(0000000000101111) as its syndrome is the same as that of (1100000011000000),
(0000000011000111) as its syndrome is the same as that of (1100000000101000),
(0000000011100101) as its syndrome is the same as that of (1100000000001010),
(0000000011101010) as its syndrome is the same as that of (1100000000000101),
(0000000011101000) as its syndrome is the same as that of (1100000000000111).

Table 1

Syndromes of Correctable error-vectors

Error vectors	Syndromes
0000000000000000	000000000000
1000000000000000	011111111001
1001000000000000	010111111001
1000100000000000	011011111001
1000010000000000	011101111001
1000001000000000	011110111001
1000000100000000	011111011001
1000000010000000	011111101001
1000000001000000	011111110001
1000000000100000	011111111101
1000000000010000	011111111011
1000000000001000	011111111000
1000000000000100	111011011101
1001100000000000	010011111001
1000110000000000	011001111001
1000011000000000	011100111001
1000001100000000	011110011001
1000000110000000	011111001001
1000000011000000	011111100001
1000000001100000	011111110101
1000000000110000	011111111111
1000000000011000	011111111010
1000000000001100	111011011100
1000000000000110	101001001111
1001010000000000	010101111001
1000101000000000	011010111001
1000010100000000	011101011001
1000001010000000	011110101001
1000000101000000	011111010001
1000000010100000	011111101101
1000000001010000	011111110011
1000000000101000	011111111100
1000000000010100	111011011111
1000000000001010	001101101010

Error vectors	Syndromes
1000000000000101	110010010100
1100000000000000	111111100010
1101000000000000	110111100010
1100100000000000	111011100010
1100010000000000	111101100010
1100001000000000	111110100010
1100000100000000	111111000010
1100000010000000	111111110010
1100000001000000	111111101010
1100000000100000	111111100110
1100000000010000	111111100000
1100000000001000	111111100011
1100000000000100	011011000110
1101100000000000	110011100010
1100110000000000	111001100010
1100011000000000	111100100010
1100001100000000	111110000010
1100000110000000	111111010010
1100000011000000	111111111010
1100000001100000	111111101110
1100000000110000	111111100100
1100000000011000	111111100001
1100000000001100	011011000111
1100000000000110	001001010100
1101010000000000	110101100010
1100101000000000	111010100010
1100010100000000	111101000010
1100001010000000	111110110010
1100000101000000	111111001010
1100000010100000	111111110110
1100000001010000	111111101000
1100000000101000	111111100111
1100000000010100	011011000100
1100000000001010	101101110001

(Contd.)

Error vectors	Syndromes
1100000000000101	010010001111
1010000000000000	001111111011
1011000000000000	000111111011
1010100000000000	001011111011
1010010000000000	001101111011
1010001000000000	001110111011
1010000100000000	001111011011
1010000010000000	001111101011
1010000001000000	001111110011
1010000000100000	001111111111
1010000000010000	001111111001
1010000000001000	001111111010
1010000000000100	101011011111
1011100000000000	000011111011
1010110000000000	001001111011
1010011000000000	001100111011
1010001100000000	001110011011
1010000110000000	001111001011
1010000011000000	001111100011
1010000001100000	001111110111
1010000000110000	001111111101
1010000000011000	001111111000
1010000000001100	101011011110
1010000000000110	111001001101
1011010000000000	000101111011
1010101000000000	001010111011
1010010100000000	001101011011
1010001010000000	001110101011
1010000101000000	001111010011
1010000010100000	001111101111
1010000001010000	001111110001
1010000000101000	001111111110
1010000000010100	101011011101
1010000000001010	011101101000
1010000000000101	100010010110
0100000000000000	100000011011
0100100000000000	100100011011

Error vectors	Syndromes
0100010000000000	100010011011
0100001000000000	100001011011
0100000100000000	100000111011
0100000010000000	100000001011
0100000001000000	100000010011
0100000000100000	100000011111
0100000000010000	100000011001
0100000000001000	100000011010
0100000000000100	000100111111
0100110000000000	100110011011
0100011000000000	100011011011
0100001100000000	100001111011
0100000110000000	100000101011
0100000011000000	100000000011
0100000001100000	100000010111
0100000000110000	100000011101
0100000000011000	100000011000
0100000000001100	000100111110
0100000000000110	010110101101
0100101000000000	100101011011
0100010100000000	100010111011
0100001010000000	100001001011
0100000101000000	100000110011
0100000010100000	100000001111
0100000001010000	100000010001
0100000000101000	100000011110
0100000000010100	000100111101
0100000000001010	110010001000
0100000000000101	001101110110
0110000000000000	110000011001
0110100000000000	110100011001
0110010000000000	110010011001
0110001000000000	110001011001
0110000100000000	110000111001
0110000010000000	110000001001
0110000001000000	110000010001
0110000000100000	110000011101

(Contd.)

Error vectors	Syndromes
0110000000010000	110000011011
0110000000001000	110000011000
0110000000000100	010100111101
0110110000000000	110110011001
0110011000000000	110011011001
0110001100000000	110001111001
0110000110000000	110000101001
0110000011000000	110000000001
0110000001100000	110000010101
0110000000110000	110000011111
0110000000011000	110000011010
0110000000001100	010100111100
0110000000000110	000110101111
0110101000000000	110101011001
0110010100000000	110010111001
0110001010000000	110001001001
0110000101000000	110000110001
0110000010100000	110000001101
0110000001010000	110000010011
0110000000101000	110000011100
0110000000010100	010100111111
0110000000001010	100010001010
0110000000000101	011101110100
0101000000000000	101000011011
0101100000000000	101100011011
0101010000000000	101010011011
0101001000000000	101001011011
0101000100000000	101000111011
0101000010000000	101000001011
0101000001000000	101000010011
0101000000100000	101000011111
0101000000010000	101000011001
0101000000001000	101000011010
0101000000000100	001100111111
0101110000000000	101110011011
0101011000000000	101011011011
0101001100000000	101001111011

Error vectors	Syndromes
0101000110000000	101000101011
0101000011000000	101000000011
0101000001100000	101000010111
0101000000110000	101000011101
0101000000011000	101000011000
0101000000001100	001100111110
0101000000000110	011110101101
0101101000000000	101101011011
0101010100000000	101010111011
0101001010000000	101001001011
0101000101000000	101000110011
0101000010100000	101000001111
0101000001010000	101000010001
0101000000101000	101000011110
0101000000010100	001100111101
0101000000001010	111010001000
0101000000000101	000101110110
0010000000000000	010000000010
0010010000000000	010010000010
0010001000000000	010001000010
0010000100000000	010000100010
0010000010000000	010000010010
0010000001000000	010000001010
0010000000100000	010000000110
0010000000010000	010000000000
0010000000001000	010000000011
0010000000000100	110100100110
0010011000000000	010011000010
0010001100000000	010001100010
0010000110000000	010000110010
0010000011000000	010000011010
0010000001100000	010000001110
0010000000110000	010000000100
0010000000011000	010000000001
0010000000001100	110100100111
0010000000000110	100110110100
0010010100000000	010010100010

(Contd.)

Error vectors	Syndromes
0010001010000000	010001010010
0010000101000000	010000101010
0010000010100000	010000010110
0010000001010000	010000001000
0010000000101000	010000000111
0010000000010100	110100100100
0010000000001010	000010010001
0010000000000101	111101101111
0011000000000000	011000000010
0011010000000000	011010000010
0011001000000000	011001000010
0011000100000000	011000100010
0011000010000000	011000010010
0011000001000000	011000001010
0011000000100000	011000000110
0011000000010000	011000000000
0011000000001000	011000000011
0011000000000100	111100100110
0011011000000000	011011000010
0011001100000000	011001100010
0011000110000000	011000110010
0011000011000000	011000011010
0011000001100000	011000001110
0011000000110000	011000000100
0011000000011000	011000000001
0011000000001100	111100100111
0011000000000110	101110110100
0011010100000000	011010100010
0011001010000000	011001010010
0011000101000000	011000101010
0011000010100000	011000010110
0011000001010000	011000001000
0011000000101000	011000000111
0011000000010100	111100100100
0011000000001010	001010010001
0011000000000101	110101101111
0010100000000000	010100000010

Error vectors	Syndromes
0010110000000000	010110000010
0010101000000000	010101000010
0010100100000000	010100100010
0010100010000000	010100010010
0010100001000000	010100001010
0010100000100000	010100000110
0010100000010000	010100000000
0010100000001000	010100000011
0010100000000100	110000100110
0010111000000000	010111000010
0010101100000000	010101100010
0010100110000000	010100110010
0010100011000000	010100011010
0010100001100000	010100001110
0010100000110000	010100000100
0010100000011000	010100000001
0010100000001100	110000100111
0010100000000110	100010110100
0010110100000000	010110100010
0010101010000000	010101010010
0010100101000000	010100101010
0010100010100000	010100010110
0010100001010000	010100001000
0010100000101000	010100000111
0010100000010100	110000100100
0010100000001010	000110010001
0010100000000101	111001101111
0001000000000000	001000000000
0001001000000000	001001000000
0001000100000000	001000100000
0001000010000000	001000010000
0001000001000000	001000001000
0001000000100000	001000000100
0001000000010000	001000000010
0001000000001000	001000000001
0001000000000100	101100100100
0001001100000000	001001100000

(Contd.)

Error vectors	Syndromes
0001000110000000	001000110000
0001000011000000	001000011000
0001000001100000	001000001100
0001000000110000	001000000110
0001000000011000	001000000011
0001000000001100	101100100101
0001000000000110	111110110110
0001001010000000	001001010000
0001000101000000	001000101000
0001000010100000	001000010100
0001000001010000	001000001010
0001000000101000	001000000101
0001000000010100	101100100110
0001000000001010	011010010011
0001000000000101	100101101101
0001100000000000	001100000000
0001101000000000	001101000000
0001100100000000	001100100000
0001100010000000	001100010000
0001100001000000	001100001000
0001100000100000	001100000100
0001100000010000	001100000010
0001100000001000	001100000001
0001100000000100	101000100100
0001101100000000	001101100000
0001100110000000	001100110000
0001100011000000	001100011000
0001100001100000	001100001100
0001100000110000	001100000110
0001100000011000	001100000011
0001100000001100	101000100101
0001100000000110	111010110110
0001101010000000	001101010000
0001100101000000	001100101000
0001100010100000	001100010100
0001100001010000	001100001010
0001100000101000	001100000101

Error vectors	Syndromes
0001100000010100	101000100110
0001100000001010	011110010011
0001100000000101	100001101101
0001010000000000	001010000000
0001011000000000	001011000000
0001010100000000	001010100000
0001010010000000	001010010000
0001010001000000	001010001000
0001010000100000	001010000100
0001010000010000	001010000010
0001010000001000	001010000001
0001010000000100	101110100100
0001011100000000	001011100000
0001010110000000	001010110000
0001010011000000	001010011000
0001010001100000	001010001100
0001010000110000	001010000110
0001010000011000	001010000011
0001010000001100	101110100101
0001010000000110	111100110110
0001011010000000	001011010000
0001010101000000	001010101000
0001010010100000	001010010100
0001010001010000	001010001010
0001010000101000	001010000101
0001010000010100	101110100110
0001010000001010	011000010011
0001010000000101	100111101101
0000100000000000	000100000000
0000100100000000	000100100000
0000100010000000	000100010000
0000100001000000	000100001000
0000100000100000	000100000100
0000100000010000	000100000010
0000100000001000	000100000001
0000100000000100	100000100100
0000100110000000	000100110000

(Contd.)

Error vectors	Syndromes
0000100011000000	000100011000
0000100001100000	000100001100
0000100000110000	000100000110
0000100000011000	000100000011
0000100000001100	100000100101
0000100000000110	110010110110
0000100101000000	000100101000
0000100010100000	000100010100
0000100001010000	000100001010
0000100000101000	000100000101
0000100000010100	100000100110
0000100000001010	010110010011
0000100000000101	101001101101
0000110000000000	000110000000
0000110100000000	000110100000
0000110010000000	000110010000
0000110001000000	000110001000
0000110000100000	000110000100
0000110000010000	000110000010
0000110000001000	000110000001
0000110000000100	100010100100
0000110110000000	000110110000
0000110011000000	000110011000
0000110001100000	000110001100
0000110000110000	000110000110
0000110000011000	000110000011
0000110000001100	100010100101
0000110000000110	110000110110
0000110101000000	000110101000
0000110010100000	000110010100
0000110001010000	000110001010
0000110000101000	000110000101
0000110000010100	100010100110
0000110000001010	010100010011
0000110000000101	101011101101
0000101000000000	000101000000
0000101100000000	000101100000

Error vectors	Syndromes
0000101010000000	000101010000
0000101001000000	000101001000
0000101000100000	000101000100
0000101000010000	000101000010
0000101000001000	000101000001
0000101000000100	100001100100
0000101110000000	000101110000
0000101011000000	000101011000
0000101001100000	000101001100
0000101000110000	000101000110
0000101000011000	000101000011
0000101000001100	100001100101
0000101000000110	110011110110
0000101101000000	000101101000
0000101010100000	000101010100
0000101001010000	000101001010
0000101000101000	000101000101
0000101000010100	100001100110
0000101000001010	010111010011
0000101000000101	101000101101
0000010000000000	000010000000
0000010010000000	000010010000
0000010001000000	000010001000
0000010000100000	000010000100
0000010000010000	000010000010
0000010000001000	000010000001
0000010000000100	100110100100
0000010011000000	000010011000
0000010001100000	000010001100
0000010000110000	000010000110
0000010000011000	000010000011
0000010000001100	100110100101
0000010000000110	110100110110
0000010010100000	000010010100
0000010001010000	000010001010
0000010000101000	000010000101
0000010000010100	100110100110

(Contd.)

Error vectors	Syndromes
0000010000001010	010000010011
0000010000000101	101111101101
0000011000000000	000011000000
0000011010000000	000011010000
0000011001000000	000011001000
0000011000100000	000011000100
0000011000010000	000011000010
0000011000001000	000011000001
0000011000000100	100111100100
0000011011000000	000011011000
0000011001100000	000011001100
0000011000110000	000011000110
0000011000011000	000011000011
0000011000001100	100111100101
0000011000000110	110101110110
0000011010100000	000011010100
0000011001010000	000011001010
0000011000101000	000011000101
0000011000010100	100111100110
0000011000001010	010001010011
0000011000000101	101110101101
0000010100000000	000010100000
0000010110000000	000010110000
0000010101000000	000010101000
0000010100100000	000010100100
0000010100010000	000010100010
0000010100001000	000010100001
0000010100000100	100110000100
0000010111000000	000010111000
0000010101100000	000010101100
0000010100110000	000010100110
0000010100011000	000010100011
0000010100001100	100110000101
0000010100000110	110100010110
0000010110100000	000010110100
0000010101010000	000010101010
0000010100101000	000010100101

Error vectors	Syndromes
0000010100010100	100110000110
0000010100001010	010000110011
0000010100000101	101111001101
0000001000000000	000001000000
0000001001000000	000001001000
0000001000100000	000001000100
0000001000010000	000001000010
0000001000001000	000001000001
0000001000000100	100101100100
0000001001100000	000001001100
0000001000110000	000001000110
0000001000011000	000001000011
0000001000001100	100101100101
0000001000000110	110111110110
0000001001010000	000001001010
0000001000101000	000001000101
0000001000010100	100101100110
0000001000001010	010011010011
0000001000000101	101100101101
0000001100000000	000001100000
0000001101000000	000001101000
0000001100100000	000001100100
0000001100010000	000001100010
0000001100001000	000001100001
0000001100000100	100101000100
0000001101100000	000001101100
0000001100110000	000001100110
0000001100011000	000001100011
0000001100001100	100101000101
0000001100000110	110111010110
0000001101010000	000001101010
0000001100101000	000001100101
0000001100010100	100101000110
0000001100001010	010011110011
0000001100000101	101100001101
0000001010000000	000001010000
0000001011000000	000001011000

(Contd.)

Error vectors	Syndromes
0000001010100000	000001010100
0000001010010000	000001010010
0000001010001000	000001010001
0000001010000100	100101110100
0000001011100000	000001011100
0000001010110000	000001010110
0000001010011000	000001010011
0000001010001100	100101110101
0000001010000110	110111100110
0000001011010000	000001011010
0000001010101000	000001010101
0000001010010100	100101110110
0000001010001010	010011000011
0000001010000101	101100111101
0000000100000000	000000100000
0000000100100000	000000100100
0000000100010000	000000100010
0000000100001000	000000100001
0000000100000100	100100000100
0000000100110000	000000100110
0000000100011000	000000100011
0000000100001100	100100000101
0000000100000110	110110010110
0000000100101000	000000100101
0000000100010100	100100000110
0000000100001010	010010110011
0000000100000101	101101001101
0000000110000000	000000110000
0000000110100000	000000110100
0000000110010000	000000110010
0000000110001000	000000110001
0000000110000100	100100010100
0000000110110000	000000110110
0000000110011000	000000110011
0000000110001100	100100010101
0000000110000110	110110000110
0000000110101000	000000110101

Error vectors	Syndromes
0000000110010100	100100010110
0000000110001010	010010100011
0000000110000101	101101011101
0000000101000000	000000101000
0000000101100000	000000101100
0000000101010000	000000101010
0000000101001000	000000101001
0000000101000100	100100001100
0000000101110000	000000101110
0000000101011000	000000101011
0000000101001100	100100001101
0000000101000110	110110011110
0000000101101000	000000101101
0000000101010100	100100001110
0000000101001010	010010111011
0000000101000101	101101000101
0000000010000000	000000010000
0000000010010000	000000010010
0000000010001000	000000010001
0000000010000100	100100110100
0000000010011000	000000010011
0000000010001100	100100110101
0000000010000110	110110100110
0000000010010100	100100110110
0000000010001010	010010000011
0000000010000101	101101111101
0000000011000000	000000011000
0000000011010000	000000011010
0000000011001000	000000011001
0000000011000100	100100111100
0000000011011000	000000011011
0000000011001100	100100111101
0000000011000110	110110101110
0000000011010100	100100111110
0000000011001010	010010001011
0000000011000101	101101110101
0000000010100000	000000010100

(Contd.)

Error vectors	Syndromes
0000000010110000	000000010110
0000000010101000	000000010101
0000000010100100	100100110000
0000000010111000	000000010111
0000000010101100	100100110001
0000000010100110	110110100010
0000000010110100	100100110010
0000000010101010	010010000111
0000000010100101	101101111001
0000000001000000	000000001000
0000000001001000	000000001001
0000000001000100	100100101100
0000000001001100	100100101101
0000000001000110	110110111110
0000000001001010	010010011011
0000000001000101	101101100101
0000000001100000	000000001100
0000000001101000	000000001101
0000000001100100	100100101000
0000000001101100	100100101001
0000000001100110	110110111010
0000000001101010	010010011111
0000000001100101	101101100001
0000000001010000	000000001010
0000000001011000	000000001011
0000000001010100	100100101110
0000000001011100	100100101111
0000000001010110	110110111100
0000000001011010	010010011001
0000000001010101	101101100111
0000000000100000	000000000100
0000000000100100	100100100000
0000000000100110	110110110010
0000000000100101	101101101001
0000000000110000	000000000110
0000000000110100	100100100010
0000000000110110	110110110000

Error vectors	Syndromes
0000000000110101	101101101011
0000000000101000	000000000101
0000000000101100	100100100001
0000000000101110	110110110011
0000000000101101	101101101000
0000000000010000	000000000010
0000000000011000	000000000011
0000000000010100	100100100110
0000000000001000	000000000001
0000000000001100	100100100101
0000000000001010	010010010011
0000000000000100	100100100100
0000000000000110	110110110110
0000000000000101	101101101101

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Stability of Additive Mappings In Generalized Normed Spaces *

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Abstract

In this paper, we introduce the concept of a Generalized normed space and prove a theorem for existence of an additive mapping in this space. We show that our results extend some of the known results in literature.

Keywords and Phrases: *Generalized normed space, Additive mappings, Cauchy function.*

1. Introduction and Preliminaries

It is well known that the Ulam's [12] question in 1940: "Under what conditions does there exist an additive mapping near an approximately additive mapping ?", is the origin of the stability problem of functional equations. Many authors have extended, generalized and improved the answer to Ulam's question such as, Hyers [4] in the context of Banach space, K. Ravi, R. Murali and M. Arunkumar [10] for quadratic functional equation, T. Aoki [1] for additive mappings and Th.M. Rassias [8] for linear mappings in 1978 by considering the unbounded Cauchy difference. It states as follows:

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Theorem 1.1. *([8]) Let $E,\ E'$ be two Banach spaces and let $\theta \in [0,\infty)$ and $p \in [0,1)$. If a function $f : E \longrightarrow E'$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta[\|x\|^p + \|y\|^p]$$

for all $x, y \in E$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p}\,\|x\|^p$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E$ then T is linear.

In the following theorem J. M. Rassias replaced the sum by the product of powers of norms.

Theorem 1.2. *([7]) Let $f : E \longrightarrow E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\,\|x\|^p\,\|y\|^p \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leq p \leq \frac{1}{2}$. Then the limit

$$L(x) = \lim_{n\longrightarrow\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \longrightarrow E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{2-2^{2p}}\,\|x\|^{2p} \tag{1.2}$$

for all $x \in E$. If $p < 0$, then the inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. If $p > \frac{1}{2}$ then the inequality (1.1) holds for $x, y \in E$ and the limit

$$A(x) = \lim_{n\longrightarrow\infty} 2^n f(\frac{x}{2^n})$$

exists for all $x \in E$ and $A : E \longrightarrow E'$ is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \frac{\epsilon}{2^{2p}-2}\,\|x\|^{2p}$$

for all $x \in E$. If in addition $f : E \longrightarrow E'$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then L is $\mathbb{R}-$linear mapping.

Also, the topic of stability of functional equations has been studied by a number of mathematicians (see [6, 2, 3, 9, 5] for more detailed information). Before giving the main results, we recall the definition of normed-binary operation and some examples and lemmas were used in [11].

Definition 1.3. ([11])A normed-binary operation is a mapping $\diamond : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ which satisfies the following conditions:

(i) $\diamond$ is associative and commutative,

(ii) $\diamond$ is continuous,

(iii) $a \diamond 0 = a$ for all $a \in [0,\infty)$,

(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,\infty)$.

Example 1.4. ([11])Let $a, b \in [0,\infty)$. Five typical examples of $\diamond$ are:

(a) $a \diamond_1 b = \max\{a, b\}$

(b) $a \diamond_2 b = \sqrt{a^2 + b^2}$

(c) $a \diamond_3 b = a + b$

(d) $a \diamond_4 b = ab + a + b$

(e) $a \diamond_5 b = (\sqrt{a} + \sqrt{b})^2$.

For $a, b \in [0,\infty)$, straight forward calculations lead to the following relations among normed binary operations giving above

$$a \diamond_1 b \leq a \diamond_2 b \leq a \diamond_3 b \leq a \diamond_4 b,$$

and

$$a \diamond_3 b \leq a \diamond_5 b.$$

The follwing lemma defines a normed binary operation exploting some properties of a self map on $[0,\infty)$.

Lemma 1.5. *([11])Let $f : [0,\infty) \longrightarrow [0,\infty)$ be a continuous, onto, and increasing map. Let $\diamond : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ be defined by*

$a \diamond b = f^{-1}(f(a) + f(b))$ for $a, b \in [0,\infty)$,

then $\diamond$ is a normed binary operation.

Example 1.6. ([11])Let $f : [0,\infty) \longrightarrow [0,\infty)$ defined by $f(x) = e^x - 1$. Then $a \diamond b = Ln(e^a + e^b - 1)$ for $a, b \in [0,\infty)$ defines a normed binary operation.

We have the following simple observations about normed binary operation.

Lemma 1.7. *(**[11]**) (i) If $r, r' \geq 0$, then $r \leq r \diamond r'$.*

(ii) If $\delta \in (0, r)$, there exist $\delta' \in (0, r)$ such that $\delta' \diamond \delta < r$.

(iii) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\delta \diamond \delta < \varepsilon$.

In this paper all vector spaces are real.

Now we are set to generalize the concept of a normed space.

Definition 1.8. Let X be vector space and $\diamond$ be a binary operation. A generalized norm on X is a function: $N : X \longrightarrow \mathbb{R}$ that satisfies the following properties:

(1) $N(x) \geq 0$ for each x in X,

(2) $N(x) = 0$ if and only if $x = 0$,

(3) $N(\alpha x) = |\alpha|^t N(x)$ for some $t \in (0, \infty)$, for each x in X and every $\alpha \in \mathbb{R}$.

(4) $N(x + y) \leq N(x) \diamond N(y)$, for each $x, y \in X$.

The $3-$*tuple* $(X, N, \diamond)$ is called a generalized normed space or a $G-$*normed* space.

Example 1.9. Let $(X, \|.\|)$ be a normed space, $a, b \in [0, \infty)$, and $x \in X$. If we define $\diamond : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$,

(i) $a \diamond b = a + b$, and N is defind by $N(x) = \|x\|$, then $(X, N, \diamond)$ is a $G-$ *normed* space for $t = 1$.

(ii) $a \diamond b = \sqrt{a^2 + b^2}$, and N is defined by $N(x) = \sqrt{\|x\|}$, then $(X, N, \diamond)$ is a $G-$ *normed* space for $t = \frac{1}{2}$.

(iii) $a \diamond b = (\sqrt{a} + \sqrt{b})^2$, and N is defined by $N(x) = \|x\|^2$, then $(X, N, \diamond)$ is a $G-$ *normed* space for $t = 2$.

Remark 1.10. From Example 1.9 (i), we see that:
every normed space is a $G-$ *normed* space.

Remark 1.11. In (3) of Definition 1.8 t is unique.

Example 1.12. Let $X = \mathbb{R}^2$, if we define $\diamond : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ by
$a \diamond b = (\sqrt[4]{a} + \sqrt[4]{b})^4$ for $a, b \in [0, \infty)$, and define $N : X \longrightarrow R$ by
$N(x, y) = x^4 + y^4$ for $x, y \in \mathbb{R}$,
then $(X, N, \diamond)$ is a $G-$ *normed* space for $t = 4$.

Definition 1.13. Let $(X, N, \diamond)$ be a $G-$ *normed* space. For $r > 0$, the ball $B_N(x, r)$ with center $x \in X$ and radius r is defined by

$$B_N(x, r) = \{y \in X : N(x - y) < r\}.$$

Definition 1.14. Let $(X, N, \diamond)$ be a $G-$ *normed* space. A subset $A \subseteq X$ is open if for every $x \in A$, there exists $r > 0$ such that $B_N(x, r) \subseteq A$.

Let τ be the set of all open subsets $A \subseteq X$. It can be verified that τ is a topology on $X,$ called a topology induced by generalized norme N.

Lemma 1.15. *Let $(X, N, \diamond)$ be a $G-normed$ space. Then*
(i) $N(ax) \leq N(x)$ for all real scalars a with $|a| \leq 1$.
(ii) if X is convex, then we get

$$N(ax + (1-a)y) \leq \ N(x) \diamond N(y)$$

for all $x, y \in X$ and every $a \in (0, 1)$.

Proof. Proof immediately follows from Definition 1.8. □

Definition 1.16. Let $(X, N, \diamond)$ be a $G-normed$ space. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$n \geq n_0 \Longrightarrow N(x_n - x) < \epsilon.$

We denote this by $N(x_n - x) \longrightarrow 0$ as $n \longrightarrow \infty$ or $\lim\limits_{n \longrightarrow \infty} x_n = x$.

Definition 1.17. Let $(X, N, \diamond)$ be a $G-normed$ space. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0,$ there exists $n_0 \in \mathbb{N}$ such that $N(x_n - x_m) < \epsilon$ for each $n, m \geq n_0$.

The generalized normed space $(X, N, \diamond)$ is said to be generalized Banach space or $G-Banach$ space if every Cauchy sequence is convergent in X.

Now we prove the following basic lemmas needed in the sequel.

Lemma 1.18. *Let $(X, N, \diamond)$ be a $G-normed$ space. If $r > 0,$ then the ball $B_N(x, r)$ is open.*

Proof. Let $y \in B_N(x, r)$, so that we have $N(x - y) < r$. Put, $N(x - y) = \delta$ then by Lemma 1.7 there exists $\delta' > 0$ such that $\delta' \diamond \delta < r$. Now, we prove that $B_N(y, \delta') \subseteq \ B_N(x, r)$. For this, let $z \in B_N(y, \delta')$. By triangle inequality we have

$$N(x - z) \leq N(x - y) \diamond N(y - z) < \delta \diamond \delta' < r.$$

This implies that

$$B_N(y, \delta') \subseteq B_N(x, r).$$

Hence $B_N(x, r)$ is an open set. □

Lemma 1.19. *Every $G-normed$ space $(X, N, \diamond)$ is a Hausdorff space.*

Proof. Let $x, y \in X$ and $x \neq y$. If we set $N(x-y) = r$ then for $0 < \delta < r$ by Lemma 1.7 there exists $0 < \delta' < r$ such that $\delta' \diamond \delta < r$. We prove that $B_N(x,\delta) \cap B_N(y,\delta') = \varnothing$. Let $z \in B_N(x,\delta) \cap B_N(y,\delta')$. Now, by triangle inequality, we get that

$$r = N(x-y) \leq N(x-z) \diamond N(z-y) < \delta \diamond \delta' < r,$$

which is a contradiction. Hence $(X, N, \diamond)$ is a Hausdorff space. □

Lemma 1.20. *Let $(X, N, \diamond)$ be a $G-$normed space, then every convergent sequence in X is Cauchy in X.*

Proof. Let $\{x_n\}$ be a sequence in X which converges to $x \in X$. For $\epsilon > 0$, by Lemma 1.7 we can choose a $\delta > 0$ such that $\delta \diamond \delta < \epsilon$. Since $x_n \longrightarrow x$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we obtain that $N(x_n - x) < \delta$.
Thus for every $n, m \geq n_0$, we have

$$N(x_n - x_m) \leq N(x_n - x) \diamond N(x - x_m) < \delta \diamond \delta < \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence. □

Lemma 1.21. *Let $(X, N, \diamond)$ be a $G-$normed space, then*
addition $+:\ X \times X \longrightarrow X$ defined by $+(x,y) = x+y$ and
scalar multiplication $\cdot:\ \mathbb{R} \times X \longrightarrow X$ defined by $\cdot(\alpha, x) = \alpha \cdot x$
are continuous.

Proof. First we prove continuity of addition. Let $x_n \longrightarrow x$, $y_n \longrightarrow y$. By Lemma 1.7 for each $\epsilon > 0$ there exists $\delta > 0$ such that $\delta \diamond \delta < \epsilon$. Also, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Longrightarrow N(x_n - x) < \delta,$$

and

$$n \geq n_0 \Longrightarrow N(y_n - y) < \delta.$$

By triangle inequality we have

$$N((x_n + y_n) - (x+y)) \leq N(x_n - x) \diamond N(y_n - y) < \delta \diamond \delta < \epsilon.$$

Now we prove that scalar multiplication is continuous. Let $\alpha_n \longrightarrow \alpha$, and $x_n \longrightarrow x$(which means that $\lim\limits_{n \longrightarrow \infty} N(x_n - x) = 0$).

Triangle inequality gives that

$$N(\alpha_n\cdot x_n-\alpha\cdot x) = N(\alpha_n\cdot(x_n-x)+(\alpha_n-\alpha)\cdot x) \leq |\alpha_n|^t N(x_n-x)\diamond|\alpha_n-\alpha|^t N(x).$$

and so

$$\limsup_{n\longrightarrow\infty} N(\alpha_n\cdot x_n-\alpha\cdot x) \leq \lim_{n\longrightarrow\infty} |\alpha_n|^t N(x_n-x) \diamond \lim_{n\longrightarrow\infty} |\alpha_n-\alpha|^t N(x) = 0.$$

Hence

$$\lim_{n\longrightarrow\infty} \alpha_n\cdot x_n = \alpha\cdot x.$$ □

Example 1.22. Let $a \diamond b = \max\{a, b\}$, then there is not any $t \in (0, \infty)$ such that

$$N(\alpha\cdot x) = |\alpha|^t \cdot N(x).$$

Because If we assume that (on contrary), there exists $t \in (0, \infty)$ and $N(\alpha\cdot x) = |\alpha|^t\cdot N(x)$. Then by taking $\alpha = 2$ we obtain:

$$|2|^t \cdot N(x) = N(2x) = N(x+x) \leq N(x) \diamond N(x) = N(x),$$

which is a contradiction.

Henceforth, we assume that the normed binary operation $\diamond$ on $[0, \infty) \times [0, \infty)$ satisfy the following properties:

(PI) : $\alpha\cdot(a \diamond b) = \alpha\cdot a \diamond \alpha\cdot b$ for every $\alpha \in \mathbb{R}^+$ and

(PII) : there exists $h \geq 0$ such that $1 \diamond 1 \diamond \cdots \diamond 1 \leq n^h$, for every $n \in N$.

In the following example, we give some normed binary operations $\diamond$ on $[0, \infty) \times [0, \infty)$ with properties (PI) and $P(II)$.

Example 1.23. Let $a \diamond b = \max\{a, b\}$ or $a \diamond b = \sqrt{a^2+b^2}$ or $a \diamond b = a + b$ or $a \diamond b = (\sqrt{a} + \sqrt{b})^2$

then in each case, $\diamond$ satisfies properties (PI) and (PII).

The next example includes a normed binary operation $\diamond$ on $[0, \infty)\times[0, \infty)$ which does not satisfy (PI) and $P(II)$ properties.

Example 1.24. Define $\diamond : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ by $a \diamond b = a + b + ab$, for $a, b \in [0, \infty)$. Obviously $\diamond$ is not have (PI) and (PII) properties.

2. Main Results

In the rest of this paper, we will assume that $(X, N', \diamond)$ is $G-normed$ space and $(Y, N, \diamond)$ is $G-Banach$ space.

Let ϕ be a function from $X \times X$ to X. A mapping $f : X \longrightarrow Y$ is called a $\phi-approximately\ Cauchy\ function$, if

$$N(f(x+y) - f(x) - f(y)) \leq N'(\phi(x,y)) \tag{2.1}$$

for all $x, y \in X$.

Example 2.1. Let $X = Y = \mathbb{R}$ and N, N' be usual norm. Let ϕ be a function from $X \times X \longrightarrow X$ defined by $\phi(x,y) = xy(\frac{x+y}{2})$.

Let mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(x) = \sin x$. Then one can easily see that f is a $\phi-approximately\ Cauchy\ function$, because:

$$\begin{aligned}
|f(x+y) - f(x) - f(y)| &= \left|-4\sin(\frac{x+y}{2})\cdot \sin\frac{x}{2}\cdot \sin\frac{y}{2}\right| \\
&\leq 4\left|\frac{x+y}{2}\cdot\frac{x}{2}\cdot\frac{y}{2}\right| \\
&= \frac{|x+y|\,|xy|}{2} = |\phi(x,y)|
\end{aligned}$$

for all $x, y \in \mathbb{R}$.

Hence $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a $\phi-approximately\ Cauchy\ function.$

In the sequel, all binary operation $\diamond$ satisfy (PI) and (PII) properties.

Theorem 2.2. *Let $\phi : X \times X \longrightarrow X$ be a function and $f : X \longrightarrow Y$ be a $\phi-approximately\ Cauchy\ function$ and for some $0 < \alpha < \frac{1}{2}$ assume that,*

$$N'(\phi(\frac{x}{2}, \frac{y}{2})) \leq N'(\alpha\phi(x,y))$$

then there exists an additive mapping $T : X \longrightarrow Y$.
Morover, if $a \diamond b \leq a + b$ for every $a, b \in [0, \infty)$, then

$$N(f(x) - T(x)) \leq \frac{\alpha^t}{1-(2\alpha)^t} N'(\phi(x,x)) \tag{2.2}$$

for all $x \in X$ and T is unique.

Proof. Since f is $\phi-approximately\ Cauchy\ function$, put $x=y$ in (2.1) to obtain

$$N(f(2x)-2f(x))\leq N'(\phi(x,x)) \quad (x\in X) \tag{2.3}$$

Replacing x by $2^{-n-1}x$ in inequality (2.3) we get

$$\begin{aligned}
N(f(\frac{x}{2^n})-2f(\frac{x}{2^{n+1}}) &\leq N'(\phi(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}))\\
&\leq N'(\alpha\phi(\frac{x}{2^n},\frac{x}{2^n}))\\
&\leq |\alpha|^t\, N'(\phi(\frac{x}{2^n},\frac{x}{2^n}))\\
&\vdots\\
&\leq |\alpha|^{nt}\, N'(\phi(x,x))
\end{aligned}$$

If set $a_n(x)=2^nf(2^{-n}x)$, we have

$$\begin{aligned}
N(a_n(x)-a_{n+1}(x)) &= N(2^nf(2^{-n}x)-2^{n+1}f(2^{-n-1}x))\\
&= 2^{nt}N(f(2^{-n}x)-2f(2^{-n-1}x))\\
&\leq |2\alpha|^{nt}\, N'(\phi(x,x))
\end{aligned}$$

Also for $n\leq m$ $(n,m\in\mathbb{N})$

$$\begin{aligned}
N(a_m(x)-a_n(x)) &\leq N(a_{n+1}(x)-a_n(x))\diamond N(a_{n+2}(x)-a_{n+1}(x))\\
&\quad \diamond\cdots\diamond N(a_m(x)-a_{m-1}(x))\\
&\leq |2\alpha|^{nt}\, N'(\phi(x,x))\diamond|2\alpha|^{(n+1)t}\, N'(\phi(x,x))\\
&\quad \diamond\cdots\diamond|2\alpha|^{(m-1)t}\, N'(\phi(x,x))\\
&\leq |2\alpha|^{nt}\, N'(\phi(x,x))(\underbrace{1\diamond 1\diamond\cdots 1}_{(m-1)-n})\\
&\leq |2\alpha|^{nt}\, N'(\phi(x,x))(\underbrace{1\diamond 1\diamond\cdots 1}_{m})\\
&\leq |2\alpha|^{nt}\, N'(\phi(x,x))\cdot m^h
\end{aligned}$$

It is easy to see that for every $m\geq n$, there exists $s>0$ such that $m\leq n^s$. Thus

$$N(a_m(x)-a_n(x))\leq|2\alpha|^{nt}\cdot n^s\cdot N'(\phi(x,x))\longrightarrow 0.$$

Which implies that $a_n(x) = 2^n f(2^{-n}x)$ is a Cauchy sequence. Since Y is $G - Banach$ space, hence, for every $x \in X$ there exists $y_v \in Y$ such that $\lim_{n \longrightarrow \infty} a_n(x) = y_v(x)$. Indeed we can define a mapping $T : X \longrightarrow Y$ by $T(x) = \lim_{n \longrightarrow \infty} a_n(x)$. That is

$$\lim_{n \longrightarrow \infty} N(2^n f(2^{-n}x) - T(x)) = 0(x \in X).$$

Now, we show that T is an additive mapping. We have

$$\begin{aligned} N(T(x+y) - Tx - Ty) \ &\leq \ N(T(x+y) - 2^n f(2^{-n}(x+y))) \\ &\quad \diamond N(2^n f(2^{-n}x) - Tx) \diamond N(2^n f(2^{-n}y - Ty) \\ &\quad \diamond N(2^n f(2^{-n}(x+y) - 2^n f(2^{-n}x) - 2^n f(2^{-n}y) \\ &\leq \ N(T(x+y) - 2^n f(2^{-n}(x+y))) \diamond N(2^n f(2^{-n}x) - Tx) \\ &\quad \diamond N(2^n f(2^{-n}y - Ty) \diamond |2\alpha|^{nt} N'(\phi(x,y)). \end{aligned}$$

As $n \longrightarrow \infty$ we get

$$N(T(x+y) - Tx - Ty) \longrightarrow 0.$$

Hence $T(x+y) = T(x) + T(y)$. Now we show that the mapping T satisfies in the inequality (2.2).

We have

$$N(f(x) - T(x)) \leq N(f(x) - 2^n f(2^{-n}x)) \diamond N(2^n f(2^{-n}x) - T(x)).$$

Since

$$\begin{aligned}
N(f(x)-2^n f(2^{-n}x)) &= N(\sum_{i=0}^{n-1} 2^i f(2^{-i}x) - 2^{i+1} f(2^{-i-1}x)) \\
&\leq N(f(x)-2f(2^{-1}x)) \diamond N(2f(2^{-1}x)-2^2 f(2^{-2}x)) \\
&\quad \diamond N(2^{n-1} f(2^{n-1}x) - 2^n f(2^{-n}x)) \\
&\leq N'(\phi(\frac{x}{2},\frac{x}{2})) \diamond 2^t N'(\phi(\frac{x}{4},\frac{x}{4})) \diamond 2^{2t} N'(\phi(\frac{x}{8},\frac{x}{8})) \\
&\quad \diamond \cdots \diamond 2^{(n-1)t} N'(\phi(\frac{x}{2^n},\frac{x}{2^n})) \\
&\leq \alpha^t N'(\phi(x,x)) \diamond 2^t \alpha^{2t} N'(\phi(x,x)) \diamond 2^{2t}\alpha^{3t} N'(\phi(x,x)) \\
&\quad \diamond \cdots \diamond 2^{(n-1)t} \alpha^{nt} N'(\phi(x,x)) \\
&\leq \alpha^t N'(\phi(x,x))(1 \diamond (2\alpha)^t \diamond (2\alpha)^{2t} \diamond \cdots \diamond (2\alpha)^{(n-1)t} \\
&\leq \alpha^t N'(\phi(x,x))(1+(2\alpha)^t + (2\alpha)^{2t} + \cdots + (2\alpha)^{(n-1)t}) \\
&\leq \frac{\alpha^t}{1-(2\alpha)^t} N'(\phi(x,x)).
\end{aligned}$$

Therefore

$$N(f(x)-T(x)) \leq \frac{\alpha^t}{1-(2\alpha)^t} N'(\phi(x,x)) \diamond N(2^n f(2^{-n}x) - T(x)).$$

As n tends to infinity we have

$$N(f(x)-T(x)) \leq \frac{\alpha^t}{1-(2\alpha)^t} N'(\phi(x,x)).$$

For uniqueness, suppose $T' : X \longrightarrow X$ is another additive mapping such that $T' \neq T$ and

$$N(f(x)-T'(x)) \leq \frac{\alpha^t}{1-(2\alpha)^t} N'(\phi(x,x))$$

for every $x \in X$.

Also, since T and T' are additive we have

$$T(x) = 2^n T(2^{-n}x) \text{ and } T'(x) = 2^n T'(2^{-n}x)$$

Hence we get

$$\begin{aligned}
N(Tx-T'x) &\leq N(2^nT(2^{-n}x)-2^nf(2^{-n}x))\diamond N(2^nf(2^{-n}x)-2^nT'(2^{-n}x))\\
&\leq 2^{nt}N(T(2^{-n}x)-f(2^{-n}x))\diamond 2^{nt}N(f(2^{-n}x)-T'(2^{-n}x))\\
&\leq 2^{nt}\frac{\alpha^t}{1-(2\alpha)^t}N'(\phi(\frac{x}{2^n},\frac{x}{2^n}))\diamond 2^{nt}\frac{\alpha^t}{1-(2\alpha)^t}N'(\phi(\frac{x}{2^n},\frac{x}{2^n}))\\
&\leq 2^{nt}\frac{\alpha^t}{1-(2\alpha)^t}\cdot\alpha^{nt}N'(\phi(x,x))\diamond 2^{nt}\frac{\alpha^t}{1-(2\alpha)^t}\cdot\alpha^{nt}N'(\phi(x,x))\\
&\leq (2\alpha)^{nt}\frac{\alpha^t}{1-(2\alpha)^t}N'(\phi(x,x))(1\diamond 1)\longrightarrow 0.
\end{aligned}$$

It follwes that $T=T'$. □

Corollary 2.3. *Let* $(X,\|\cdot\|_2)$ *be normed space and* $(Y,\|\cdot\|_1)$ *be Banach Space. Let* ϕ *be a function from* $X\times X$ *to* X, *and mapping* $f:X\longrightarrow Y$ *be a* $\phi-$ *approximately Cauchy function. If for some* $0<\alpha<\frac{1}{2}$ *assume that,*

$$\left\|\phi(\frac{x}{2},\frac{y}{2})\right\|_2\leq\|\alpha\phi(x,y)\|_2$$

then there exists a unique additive mapping $T:X\longrightarrow Y$ *such that*

$$\|f(x)-T(x)\|_1\leq\frac{\alpha}{1-2\alpha}\|\phi(x,x)\|_2$$

for all $x\in X$.

Proof. If we take $(X,\|\cdot\|_2)=(X,N',\diamond)$ and $(Y,\|\cdot\|_1)=(Y,N,\diamond)$ we get the proof by Theorem 2.2 and Remark 1.10. □

Example 2.4. Let $X=Y=\mathbb{R}$ and N,N' be usual norm. Let ϕ be a function from $\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ defined by $\phi(x,y)=xy(\frac{x+y}{2})$. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ is defind by $f(x)=\sin x$. By Example 2.1 f is a $\phi-$ *approximately Cauchy function* and

$$\left|\phi(\frac{x}{2},\frac{y}{2})\right|\leq|\alpha\phi(x,y)|\quad\text{for }\frac{1}{8}\leq\alpha<\frac{1}{2}.$$

Hence all conditions of Corollary 2.3 are hold. So there exists a unique additive mapping $T:\mathbb{R}\longrightarrow\mathbb{R}$ such that

$$|f(x)-T(x)|\leq\frac{\alpha}{1-2\alpha}\left|x^3\right|.$$

where

$$T(x) = \lim_{n \longrightarrow \infty} 2^n f(2^{-n}x) = \lim_{n \longrightarrow \infty} 2^n \sin(\frac{x}{2^n}) = x.$$

And

$$|f(x) - T(x)| = |\sin x - x| \leq \frac{\alpha}{1-2\alpha} \left|x^3\right|.$$

We show that corollary (2.3) extends the theorems 1.1 ([8]) and 1.2 ([7]).

Corollary 2.5. *Let X, X' be two Banach spaces and let $\theta \in [0, \infty)$. If a function $f : X \longrightarrow X'$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta[\|x\|^p + \|y\|^p] \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $T : X \longrightarrow X'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2^p - 2} \|x\|^p, \; \textit{for} \;\; p > 1. \tag{2.5}$$

And

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p, \quad \textit{for} \;\; 0 < p < 1. \tag{2.6}$$

Proof. We show that, if set

$$\phi(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \dfrac{\theta}{\|x\| + \|y\|} \cdot (\|x\|^p + \|y\|^p)(x+y) & (x,y) \neq (0,0) \end{cases}$$

Then, all conditions of corollary (2.3) are established. Case of $(x, y) = (0, 0)$ is obviously. In case of $(x, y) \neq (0, 0)$ we have

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \|\phi(x,y)\| \\ &= \frac{\theta}{\|x\| + \|y\|} \cdot (\|x\|^p + \|y\|^p) \|x+y\| \\ &\leq \theta \cdot (\|x\|^p + \|y\|^p) \end{aligned}$$

Also we have

$$\begin{aligned} \left\|\phi(\frac{x}{2}, \frac{y}{2})\right\| &= \frac{\theta}{\|x\| + \|y\|} \cdot (\|x\|^p + \|y\|^p) \cdot \|x+y\| \cdot \frac{1}{2^p} \\ &= \theta \cdot (\|x\|^p + \|y\|^p) \cdot \|x+y\| \cdot \frac{1}{2^p} \\ &= \frac{1}{2^p} \|\phi(x,y)\| \end{aligned}$$

i) Let $p > 1$. If set $\alpha = \frac{1}{2^p}$, then $\alpha < \frac{1}{2}$ and by corollary (2.3) there exists an additive mapping $T : X \longrightarrow Y$ such that

$$\begin{aligned}
\|f(x) - T(x)\| &\leq \frac{\alpha}{1-2\alpha}\|\phi(x,x)\| \\
&= \frac{\frac{1}{2^p}}{1-\frac{2}{2^p}} \cdot \frac{\theta}{2\|x\|} \cdot 2\|x\|^p . \|2x\| \\
&= \frac{2\theta}{2^p - 2}\|x\|^p .
\end{aligned}$$

ii) Let $0 < p < 1$. If set $\alpha = \frac{2^p}{4}$, then $\alpha < \frac{1}{2}$ and by corollary (2.3) there exists an additive mapping $T : X \longrightarrow Y$ such that

$$\begin{aligned}
\|f(x) - T(x)\| &\leq \frac{\alpha}{1-2\alpha}\|\phi(x,x)\| \\
&= \frac{\frac{2^p}{4}}{1-2\cdot\frac{2^p}{4}} \cdot \frac{\theta}{2\|x\|} \cdot 2\|x\|^p \cdot \|2x\| \\
&= \frac{2^p}{4-2.2^p} \cdot 2\theta\|x\|^p \\
&= \frac{2^{p-1}}{2-2^p} \cdot 2\theta\|x\|^p \leq \frac{2\theta}{2-2^p}\|x\|^p . \quad \text{(By considering } p < 1.\text{)}
\end{aligned}$$

□

Corollary 2.6. *Let $f : X \longrightarrow X'$ be a mapping from a normed vector space X into a Banach space X' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\|x\|^p\|y\|^p$$

for all $x, y \in X$, where $\epsilon > 0$.
If $p > \frac{1}{2}$, then there exists $T : X \longrightarrow X'$ such that

$$T(x) = \lim_{n\longrightarrow\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$ *and* $T : X \longrightarrow X'$ *is the unique additive mapping which satisfies*

$$\|f(x) - T(x)\| \leq \frac{\epsilon}{2^{2p} - 2} \|x\|^{2p}.$$

If $0 < p < \frac{1}{2}$, *then there exists* $T : X \longrightarrow X'$ *such that*

$$T(x) = \lim_{n \longrightarrow \infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$ *and* $T : X \longrightarrow X'$ *is the unique additive mapping which satisfies*

$$\|f(x) - T(x)\| \leq \frac{\epsilon}{2 - 2^{2p}} \|x\|^{2p}.$$

Proof. We show that if set

$$\phi(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \frac{\epsilon}{\|x\|^{1-p} + \|y\|^{1-p}}(x\|y\|^p + y\|x\|^p) & (x,y) \neq (0,0) \end{cases}$$

Then all conditions of corollary (2.3) are established.

Case of $(x,y) = (0,0)$ is obviously. In case of $(x,y) \neq (0,0)$ we have

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \|\phi(x,y)\| \\ &\leq \frac{\epsilon}{\|x\|^{1-p} + \|y\|^{1-p}} \cdot \|(x\|y\|^p + y\|x\|^p)\| \\ &\leq \frac{\epsilon}{\|x\|^{1-p} + \|y\|^{1-p}} \cdot \|x\|^p \|y\|^p \cdot (\|x\|^{1-p} + \|y\|^{1-p}) \\ &\leq \epsilon \|x\|^p \|y\|^p. \end{aligned}$$

Moreover

$$\begin{aligned} \left\|\phi(\frac{x}{2}, \frac{y}{2})\right\| &= \frac{\epsilon 2^{1-p}}{\|x\|^{1-p} + \|y\|^{1-p}} \cdot \|(x\|y\|^p + y\|x\|^p)\| \cdot \frac{1}{2^{p+1}} \\ &= \frac{1}{2^{2p}} \cdot \frac{\epsilon}{\|x\|^{1-p} + \|y\|^{1-p}} \cdot \|(x\|y\|^p + y\|x\|^p)\| \\ &= \frac{1}{2^{2p}} \cdot \|\phi(x,y)\|. \end{aligned}$$

i) Let $p > \frac{1}{2}$. If set $\alpha = \frac{1}{2^{2p}}$, then $\alpha < \frac{1}{2}$ and by corollary (2.3) there exists an additive mapping $T : X \longrightarrow Y$ such that

$$\begin{aligned}
\|f(x) - T(x)\| &\leq \frac{\alpha}{1-2\alpha} \|\phi(x,x)\| \\
&= \frac{\frac{1}{2^{2p}}}{1 - \frac{1}{2^{2p}}} \cdot \frac{\epsilon}{2\|x\|^{1-p}} \cdot 2\|x\|^{1+p} \\
&= \frac{\epsilon}{2^{2p} - 2} \|x\|^{2p}.
\end{aligned}$$

ii) Let $0 < p < \frac{1}{2}$. If set $\alpha = \frac{2^{2p}}{4}$, then $\alpha < \frac{1}{2}$ and by corollary (2.3) there exists an additive mapping $T : X \longrightarrow Y$ such that

$$\begin{aligned}
\|f(x) - T(x)\| &\leq \frac{\alpha}{1-2\alpha} \|\phi(x,x)\| \\
&= \frac{\frac{2^{2p}}{4}}{1 - \frac{2^{2p}}{4}} \cdot \epsilon \|x\|^{2p} \\
&= \frac{2^{2p}}{4 - 2.2^{2p}} \cdot \epsilon \|x\|^{2p} \\
&= \frac{2^{2p-1}}{2 - 2^{2p}} \cdot \epsilon \|x\|^{2p} \\
&\leq \frac{\epsilon}{2 - 2^{2p}} \|x\|^{2p} \quad (\text{By considering } p < \frac{1}{2})
\end{aligned}$$

$\square$

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Relations for Marginal and Joint Moment Generating Functions of Extended Type I Generalized Logistic Distribution based on Lower Generalized Order Statistics and Characterization *

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Abstract

In this study we give exact expressions and some recurrence relations for marginal and joint moment generating functions of lower generalized order statistics from extended type I generalized logistic distribution. The results for order statistics and lower record values are deduced from the relations derived. Further two characterization Theorems of this distribution has been considered on using conditional expectation and recurrence relations for marginal moment generating functions of the lower generalized order statistics is presented.

Keywords and Phrases: *Lower generalized order statistics, Order statistics, Lower record values, Extended type I generalized logistic distribution, Marginal and joint moment generating function, Recurrence relations and characterization.*

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1. Introduction

Kamps [18] introduced the concept of generalized order statistics (gos). It is know that order statistics, upper record values and sequential order statistics are special cases of gos. In this paper we will consider the lower generalized order statistics ($lgos$). Which is given as

Let $n \in N$, $k \geq 1$, $m \in \Re$, be the parameters such that

$$\gamma_r = k + (n-r)(m+1) > 0, \quad \text{for all } \ 1 \leq r \leq n.$$

Then $X^*(1,n,m,k), \ldots, X^*(n,n,m,k)$ are n $lgos$ from an absolutely continuous distribution function (df) $F(x)$ with the corresponding probability density function (pdf) $f(x)$ if their joint pdf has the form

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}[F(x_i)]^{m_i} f(x_i)\Big)[F(x_n)]^{k-1} f(x_n) \tag{1.1}$$

for $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.

The marginal pdf of the $r-$th $lgos$, $X^*(r,n,m,k)$ is

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \tag{1.2}$$

and the joint pdf of $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$, $1 \leq r < s \leq n$ is expressed from (1.1) as

$$f_{X^*(r,n,m,k),X^*(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}(F(x))$$

$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y), \quad x > y, \tag{1.3}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i \ , \quad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}x^{m+1}, & m \neq -1 \\ -lnx, & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1).$$

We shall also take $X^*(0, n, m, k) = 0$. If $m = 0$, $k = 1$, then $X^*(r, n, m, k)$ reduced to the $(n - r + 1)-$th order statistics, $X_{n-r+1:n}$ from the sample $X_1, X_2, \ldots, X_n$ and when $m = -1$, then $X^*(r, n, m, k)$ reduced to the $r-$ th lower k record value (Pawlas and Szynal, [14]). The work of Burkschat *et al.* [11] may also refer for lower generalized order statistics.

Saran and Pandey [7] and Khan *et al.* [16] have established recurrence relations for marginal and joint moment generating functions of *lgos* from power function and generalized exponential distributions. Ahsanullah and Raqab [10], Raqab and Ahsanullah [12, 13] and Kumar [3] have established recurrence relations for moment generating functions of record values from Pareto, Gumble, power function, extreme value and generalized logistic distributions. Recurrence relations for marginal and joint moment generating functions of from power function and Erlange-truncated exponential distribution are derived by Saran and Singh [8] and Kumar *et al.* [4]. Al-Hussaini *et al.* [5, 6] have established recurrence relations for conditional and joint moment generating functions of *gos* based on mixed population, respectively. Khan *et al.* [15] have established explicit expressions and some recurrence relations for moment generating function of generalized order statistics from Gompertz distribution. Kamps [19] investigated the importance of recurrence relations of order statistics in characterization.
In the present study, we have established exact expressions and some recurrence relations for marginal and joint moment generating functions of *lgos* from extended type I generalized logistic distribution. Results for order statistics and lower record values are deduced as special cases and a characterization of extended type I generalized logistic distribution has been obtained on using conditional expectation and recurrence relation for marginal moment generating functions.
A random variable X is said to have extended type I generalized logistic distribution if its *pdf* is of the form

$$f(x) = \frac{\alpha\lambda^{\alpha}e^{-x}}{(\lambda + e^{-x})^{\alpha+1}}, \quad -\infty < x < \infty, \quad \alpha, \lambda > 0 \tag{1.4}$$

and the corresponding *df* is

$$F(x) = \Big(\frac{\lambda}{\lambda + e^{-x}}\Big)^{\alpha}, \quad -\infty < x < \infty, \quad \alpha, \lambda > 0 \tag{1.5}$$

When $\alpha = \lambda = 1$, we have the ordinary logistic distribution and when $\lambda = 1$, we have the type I generalized logistic distribution.
For more details on this distribution and its applications one may refer to Olapade [1, 2].

2. Relations for Marginal Moment Generating Function

Note that for extended type I generalized logistic distribution defined in (1.4)

$$\alpha F(x) = (1 + \lambda e^x) f(x). \tag{2.1}$$

The relation in (2.1) will be used to derive some recurrence relations for the moment generating function of $lgos$ from extended type I generalized logistic distribution.
Let us denote the marginal moment generating functions of $X^*(r, n, m, k)$ by $M_{X^*(r,n,m,k)}(t)$ and its $j-$th derivative by $M^{(j)}_{X^*(r,n,m,k)}(t)$.
We shall first establish the explicit expressions for $M_{X^*(r,n,m,k)}(t)$. Using (1.2), we obtain when $m \neq -1$

$$\begin{aligned} M_{X^*(r,n,m,k)}(t) &= \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx \\ &= \frac{C_{r-1}}{(r-1)!} I_j(\gamma_r - 1, r - 1) \end{aligned} \tag{2.2}$$

where

$$I_j(a, b) = \int_{-\infty}^{\infty} e^{tx} [F(x)]^a f(x) g_m^b(F(x)) dx \tag{2.3}$$

Further, on using the binomial expansion, we can rewrite (2.3) as

$$I_j(a, b) = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \left(\begin{array}{c} b \\ u \end{array} \right) \int_{-\infty}^{\infty} e^{tx} [F(x)]^{a+(m+1)u} f(x) dx \tag{2.4}$$

Making the substitution $z = [F(x)]^{1/\alpha}$, (2.4) reduces to

$$I_j(a, b) = \frac{\alpha}{\lambda^t (m+1)^b} \sum_{u=0}^{b} (-1)^u \left(\begin{array}{c} b \\ u \end{array} \right) \int_0^1 (1 - z)^{-t} z^{\alpha(a+(m+1)u+1)+t-1} dz.$$

On using the Maclaurine series expansion $(1-z)^{-t} = \sum_{p=0}^{\infty} \frac{(t)_{(p)} z^p}{p!}$, where

$$(t)_p = \begin{cases} t(t+1)\ldots(t+p-1), & p=1,2,\ldots \\ 1, & p=0 \end{cases}$$

and integrating the resulting expression, we obtain

$$I_j(a,b) = \frac{\alpha}{\lambda^t (m+1)^b} \sum_{p=0}^{\infty} \sum_{u=0}^{b} (-1)^u \binom{b}{u} \frac{(t)_{(p)}}{p![\alpha(a+(m+1)u+1)+t+p]}, \ t \neq 0 \tag{2.5}$$

and when $m=-1$ that

$$I_j(a,b) = \frac{0}{0}, \qquad as \qquad \sum_{u=0}^{b} (-1)^u \binom{b}{u} = 0.$$

Since $I_j(a,b)$ is of the form $\frac{0}{0}$ at $m=-1$, therefore we have

$$I_j(a,b) = \frac{\alpha}{\lambda^t (m+1)^b} \sum_{p=0}^{\infty} \sum_{u=0}^{b} (-1)^u \binom{b}{u} \frac{(t)_{(p)}}{p![\alpha(a+(m+1)u+1)+t+p]}. \tag{2.6}$$

Differentiating numerator and denominator of (2.6) b times with respect to m, we get

$$I_j(a,b) = \frac{\alpha}{\lambda^t} \sum_{p=0}^{\infty} \sum_{u=0}^{b} (-1)^{u+b} \binom{b}{u} \frac{(t)_{(p)} \alpha^b u^b}{p![\alpha(a+(m+1)u+1)+t+p]^{b+1}}.$$

On applying L' Hospital rule, we have

$$\lim_{m\to -1} I_j(a,b) = \frac{\alpha^{b+1}}{\lambda^t} \sum_{p=0}^{\infty} \frac{(t)_{(p)}}{p![\alpha(a+1)+t+p]^{b+1}} \sum_{u=0}^{b} (-1)^{u+b} \binom{b}{u} u^b, \quad b>0.$$

But for all integers $n \geq 0$ and for all real numbers x, we have Ruiz [17]

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)^n = n!. \tag{2.7}$$

Therefore,

$$\sum_{u=0}^{b} (-1)^{u+b} \binom{b}{u} u^b = b!.$$

Hence

$$\lim_{m\to -1} I_j(a,b) = \frac{b!\alpha^{b+1}}{\lambda^t}\sum_{p=0}^{\infty}\frac{(t)_{(p)}}{p![\alpha(a+1)+t+p]^{b+1}}. \tag{2.8}$$

Now substituting the above expressions for $I_j(\gamma_r - 1, r-1)$ in (2.2) and simplifying, we obtain when $m \neq -1$ that

$$M_{X^*(r,n,m,k)}(t) = \frac{\alpha C_{r-1}}{\lambda^t (r-1)!(m+1)^{r-1}}\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u}\frac{(t)_{(p)}}{p!(\alpha\gamma_{r-u}+t+p)}. \tag{2.9}$$

and when $m = -1$ that

$$M_{X^*(r,n,-1,k)}(t) = \frac{(\alpha k)^r}{\lambda^t}\sum_{p=0}^{\infty}\frac{(t)_{(p)}}{p!(\alpha k+t+p)^r}, \quad t \neq 0. \tag{2.10}$$

Applying D' Alembert's Ratio test for convergence, it can easily be seen that $M_{X^*(r,n,-1,k)}(t)$ exist $\forall\ t, \quad -\infty < t < \infty$ and is analytic in $\forall\ t$.
Differentiating (2.9) and (2.10) and evaluating at $t = 0$, we get the mean of the $r-$th *lgos*.

Special cases

i) Putting $m = 0$, $k = 1$ in (2.9), the explicit formula for marginal moment generating functions of order statistics of the extended type I generalized logistic distribution can be obtained as

$$M_{X_{n-r+1:n}}(t) = \frac{\alpha C_{r:n}}{\lambda^t}\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u}\frac{(t)_{(p)}}{p![\alpha(n-r+1+u)+t+p]}.$$

That is

$$M_{X_{r:n}}(t) = \frac{\alpha C_{r:n}}{\lambda^t}\sum_{p=0}^{\infty}\sum_{u=0}^{n-r}(-1)^u \binom{n-r}{u}\frac{(t)_{(p)}}{p![\alpha(r+u)+t+p]},$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}.$$

ii) Setting $k = 1$ in (2.10), we get the explicit expression for marginal moment generating function of lower record values from extended type I generalized logistic distribution can be obtained as

$$M_{X_{L(r)}}(t) = \frac{\alpha^r}{\lambda^t} \sum_{p=0}^{\infty} \frac{(t)_{(p)}}{p![\alpha + t + p]^r}, \quad t \neq 0.$$

A recurrence relation for marginal moment generating function for *lgos* from *df* (1.5) can be obtained in the following theorem.

Theorem 2.1. *For the distribution given in (1.5) and for* $2 \leq r \leq n, \ n \geq 2$ *and* $k = 1, 2...$

$$\Big(1 + \frac{t}{\alpha\gamma_r}\Big) M^{(j)}_{X^*(r,n,m,k)}(t) = M^{(j)}_{X^*(r-1,n,m,k)}(t) - \frac{j}{\alpha\gamma_r} M^{(j-1)}_{X^*(r,n,m,k)}(t)$$
$$- \frac{\lambda}{\alpha\gamma_r}\Big\{ t M^{(j)}_{X^*(r,n,m,k)}(t+1) + j M^{(j-1)}_{X^*(r,n,m,k)}(t+1)\Big\}. \tag{2.11}$$

Proof. From (1.2), We have

$$M_{X^*(r,n,m,k)}(t) = \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx. \tag{2.12}$$

Integrating by parts treating $[F(x)]^{\gamma_r-1} f(x)$ for integration and rest of the integrand for differentiation, we get

$$M_{X^*(r,n,m,k)}(t) = M_{X^*(r-1,n,m,k)}(t) - \frac{tC_{r-1}}{\gamma_r (r-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{\gamma_r} g_m^{r-1}(F(x)) dx,$$

the constant of integration vanishes since the integral considered in (2.12) is a definite integral. On using (2.1), we obtain

$$M_{X^*(r,n,m,k)}(t) = M_{X^*(r-1,n,m,k)}(t)$$
$$- \frac{tC_{r-1}}{\gamma_r (r-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{\gamma_r - 1} \Big\{ \frac{(1 + \lambda e^x)}{\alpha} f(x) \Big\} g_m^{r-1}(F(x)) dx$$
$$= M_{X^*(r-1,n,m,k)}(t) - \frac{t}{\alpha\gamma_r}\Big\{ M_{X^*(r,n,m,k)}(t) - \lambda M_{X^*(r,n,m,k)}(t+1)\Big\}. \tag{2.13}$$

Differentiating both the sides of (2.13) j times with respect to t, we get

$$M^{(j)}_{X^*(r,n,m,k)}(t) = M^{(j)}_{X^*(r-1,n,m,k)}(t) - \frac{t}{\alpha\gamma_r} M^{(j)}_{X^*(r,n,m,k)}(t)$$

$$-\frac{j}{\alpha\gamma_r} M^{(j-1)}_{X^*(r,n,m,k)}(t) - \frac{\lambda t}{\alpha\gamma_r} M^{(j)}_{X^*(r,n,m,k)}(t+1) - \frac{\lambda j}{\alpha\gamma_r} M^{(j-1)}_{X^*(r,n,m,k)}(t+1).$$

The recurrence relation in equation (2.11) is derived simply by rewriting the above equation.
By differentiating both sides of equation (2.11) with respect to t and then setting $t = 0$, we obtain the recurrence relations for moments of $lgos$ from extended type I generalized logistic distribution in the form

$$E[X^{*j}(r,n,m,k)] = E[X^{*j}(r-1,n,m,k)]$$

$$-\frac{j}{\alpha\gamma_r}\Big\{E[X^{*j-1}(r,n,m,k)] + \lambda E[\phi(X^*(r,n,m,k))]\Big\}, \tag{2.14}$$

where

$$\phi(x) = x^{j-1}e^x.$$

Remark 2.1. Putting $m = 0$, $k = 1$ in (2.11), we obtain the recurrence relation for marginal moment generating function of order statistics for extended type I generalized logistic distribution in the form

$$\Big(1 + \frac{t}{\alpha(n-r+1)}\Big) M^{(j)}_{X_{n-r+1:n}}(t) = M^{(j)}_{X_{n-r+2:n}}(t) - \frac{j}{\alpha(n-r+1)} M^{(j-1)}_{X_{n-r+1:n}}(t)$$

$$-\frac{\lambda}{\alpha(n-r+1)}\Big\{t M^{(j)}_{X_{n-r+1:n}}(t+1) + j M^{(j-1)}_{X_{n-r+1:n}}(t+1)\Big\}.$$

Replacing $(n-r+1)$ by $r-1$, we have

$$M^{(j)}_{X_{r:n}}(t) = \Big(1 + \frac{t}{\alpha(r-1)}\Big) M^{(j)}_{X_{r-1:n}}(t) + \frac{j}{\alpha(r-1)} M^{(j)}_{X_{r-1:n}}(t)$$

$$+\frac{\lambda}{\alpha(r-1)}\Big\{t M^{(j)}_{X_{r-1:n}}(t+1) + j M^{(j-1)}_{X_{r-1:n}}(t+1)\Big\}.$$

Remark 2.2. Setting $m=-1$ and $k\geq 1$, in Theorem 2.1, we get a recurrence relation for marginal moment generating function of lower k record values for extended type I generalized logistic distribution in the form

$$\Big(1+\frac{t}{\alpha k}\Big)M^{(j)}_{X^*(r,n,-1,k)}(t)=M^{(j)}_{X^*(r-1,n,-1,k)}(t)-\frac{j}{\alpha k}M^{(j-1)}_{X^*(r,n,-1,k)}(t)$$
$$-\frac{\lambda}{\alpha k}\Big\{tM^{(j)}_{X^*(r,n,-1,k)}(t+1)+jM^{(j-1)}_{X^*(r-1,n,-1,k)}(t+1)\Big\}.$$

3. Relations for Joint Moment Generating Function

On using (1.3) and binomial expansion, the explicit expression for the joint moment generating of $lgos$ $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$ $1\leq r<s\leq n$, can be obtained when $m\neq -1$ as

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$
$$\times\sum_{a=o}^{r-1}\sum_{b=0}^{s-r-1}(-1)^{a+b}\binom{r-1}{a}\binom{s-r-1}{b}$$
$$\times\int_{-\infty}^{\infty}e^{t_1x}[F(x)]^{(s-r+a)(m+1)-1}f(x)I(x)dx,\quad x>y \tag{3.1}$$

where

$$I(x)=\int_{-\infty}^{x}e^{t_2y}[F(y)]^{\gamma_{s-b}-1}f(y)dy. \tag{3.2}$$

By setting $z=[F(y)]^{1/\alpha}$ in (3.2), we obtain

$$I(x)=\frac{\alpha}{\lambda^{t_2}}\sum_{p=0}^{\infty}\frac{(t_2)_{(p)}[F(x)]^{\gamma_{s-b}+(p+t_2)/\alpha}}{p!(\alpha\gamma_{s-b}+p+t_2)}.$$

On substituting the above expression of $I(x)$ in (3.1), we find that

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2)=\frac{\alpha C_{s-1}}{\lambda^{t_2}(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times \sum_{p=0}^{\infty} \sum_{a=o}^{r-1} \sum_{b=0}^{s-r-1} (-1)^{a+b} \binom{r-1}{a} \binom{s-r-1}{b}$$

$$\times \frac{(t_2)_{(p)}}{p!(\alpha\gamma_{s-b}+p+t_2)} \int_{-\infty}^{\infty} e^{t_1 x} [F(x)]^{\gamma_{r-a}-1+(p+t_2)/\alpha} f(x) dx$$

$$= \frac{\alpha^2 C_{s-1}}{\lambda^{t_1+t_2}(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{a=o}^{r-1} \sum_{b=0}^{s-r-1} (-1)^{a+b} \binom{r-1}{a}$$

$$\times \binom{s-r-1}{b} \frac{(t_2)_{(p)}(t_1)_{(q)}}{p!q!(\alpha\gamma_{s-b}+p+t_2)(\alpha\gamma_{r-a}+p+q+t_1+t_2)} \tag{3.3}$$

and for $s = r+1$

$$M_{X^*(r,n,m,k),X^*(r+1,n,m,k)}(t_1,t_2)$$

$$= \frac{\alpha^2 C_r}{\lambda^{t_1+t_2}(r-1)!(m+1)^{r-1}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{a=o}^{r-1} (-1)^a \binom{r-1}{a}$$

$$\times \frac{(t_2)_{(p)}(t_1)_{(q)}}{p!q!(\alpha\gamma_{r+1}+p+t_2)(\alpha\gamma_{r-a}+p+q+t_1+t_2)}. \tag{3.4}$$

At $m = -1$, (3.3) is of the form $\frac{0}{0}$ as $\sum_{a=0}^{r-1}(-1)^a \binom{r-1}{a} = 0$.
Therefore applying L' Hospital rule and using (2.7), we have

$$M_{X^*(r,n,-1,k),X^*(s,n,-1,k)}(t_1,t_2)$$

$$= \frac{(\alpha k)^s}{\lambda^{t_1+t_2}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(t_2)_{(p)}(t_1)_{(q)}}{p!q!(\alpha k+p+t_2)^{s-r}(\alpha k+p+q+t_1+t_2)^r} \tag{3.5}$$

and for $s = r+1$

$$M_{X^*(r,n,-1,k),X^*(r+1,n,-1,k)}(t_1,t_2)$$

$$= \frac{(\alpha k)^{r+1}}{\lambda^{t_1+t_2}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(t_2)_{(p)}(t_1)_{(q)}}{p!q!(\alpha k+p+t_2)(\alpha k+p+q+t_1+t_2)^r}. \tag{3.6}$$

Differentiating (3.3), (3.4), (3.5) and (3.6) and evaluating at $t_1 = t_2 = 0$, we get the mean of the $r-$th *lgos*.

Special cases

i) Putting $m = 0, \quad k = 1$, in (3.3), the explicit formula for joint moment generating functions of order statistics for extended type I generalized logistic distribution can be obtained as

$$M_{X_{n-r+1,n-s+1:n}}(t_1,t_2)$$

$$= \frac{\alpha^2 C_{r,s:n}}{\lambda^{t_1+t_2}} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{a=o}^{r-1}\sum_{b=0}^{s-r-1}(-1)^{a+b}\begin{pmatrix} r-1 \\ a \end{pmatrix}\begin{pmatrix} s-r-1 \\ b \end{pmatrix}$$

$$\times \frac{(t_2)_{(p)}(t_1)_{(q)}}{p!q![\alpha(n-s+1+b)+p+t_2][\alpha(n-r+1+a)+p+q+t_1+t_2]}.$$

That is

$$M_{X_{r,s:n}}(t_1,t_2) = \frac{\alpha^2 C_{r,s:n}}{\lambda^{t_1+t_2}} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{a=o}^{r-1}\sum_{b=0}^{s-r-1}(-1)^{a+b}\begin{pmatrix} r-1 \\ a \end{pmatrix}\begin{pmatrix} s-r-1 \\ b \end{pmatrix}$$

$$\times \frac{(t_2)_{(p)}(t_1)_{(q)}}{p!q![\alpha(r+b)+p+t_2][\alpha(s+a)+p+q+t_1+t_2]},$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

ii) Setting $k = 1$ in (3.5), we get the explicit expression for joint moment generating function of lower record value for extended type I generalized logistic distribution can be obtained as

$$M_{X_{L(r)},X_{L(s)}}(t_1,t_2) = \frac{\alpha^s}{\lambda^{t_1+t_2}} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(t_2)_{(p)}(t_1)_{(q)}}{p!q!(\alpha+p+t_2)^{s-r}(\alpha+p+q+t_1+t_2)^r}.$$

Making use of (2.1), we can derive the recurrence relations for joint moment generating function of *lgos* from (1.5).

Theorem 3.1. *For the distribution given in (1.5) and for* $1 \le r < s \le n$, $n \ge 2$ *and* $k = 1, 2...$

$$\left(1 + \frac{t_2}{\alpha\gamma_s}\right) M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = M^{(i,j)}_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1,t_2)$$

$$-\frac{j}{\alpha\gamma_s} M^{(i,j-1)}_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1,t_2)$$

$$-\frac{\lambda}{\alpha\gamma_s}\Big\{t_2 M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1)$$

$$+jM^{(i,j-1)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1)\Big\}. \tag{3.7}$$

Proof. Using (1.3), the joint moment generating function of $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$ is given by

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2)$$

$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}[F(x)]^m f(x) g_m^{r-1}(F(x))G(x)dx \tag{3.8}$$

where

$$G(x) = \int_{-\infty}^{x} e^{t_1x+t_2y}[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dy.$$

Solving the integral in $G(x)$ by parts and substituting the resulting expression in (3.8), we get

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = M_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1,t_2)$$

$$-\frac{t_2 C_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x} e^{t_1x+t_2y}[F(x)]^m f(x)$$

$$\times g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s}dydx,$$

the constant of integration vanishes since the integral in $G(x)$ is a definite integral. On using the relation (2.1), we obtain

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = M_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1,t_2)$$

$$-\frac{t_2}{\alpha\gamma_s} M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2)$$

$$-\frac{\lambda t_2}{\alpha\gamma_s} M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1). \tag{3.9}$$

Differentiating both the sides of (3.9) i times with respect to t_1 and then j times with respect to t_2, we get

$$M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = M^{(i,j)}_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1,t_2)$$

$$-\frac{t_2}{\alpha\gamma_s}M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2)$$

$$-\frac{j}{\alpha\gamma_s}M^{(i,j-1)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2)$$

$$-\frac{\lambda t_2}{\alpha\gamma_s}M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1)$$

$$-\frac{j\lambda}{\alpha\gamma_s}M^{(i,j-1)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1),$$

which, when rewritten gives the recurrence relation in (3.7).
One can also note that Theorem 2.1 can be deduced from Theorem 3.1 by letting t_1 tends to zero.
By differentiating both sides of equation (3.7) with respect to t_1, t_2 and then setting $t_1 = t_2 = 0$, we obtain the recurrence relations for product moments of *lgos* from extended type I generalized logistic distribution in the form

$$E[X^{*i}(r,n,m,k),X^{*j}(s,n,m,k)] = E[X^{*i}(r,n,m,k),X^{*j}(s-1,n,m,k)]$$

$$-\frac{j}{\alpha\gamma_s}\Big\{E[X^{*i}(r,n,m,k),X^{*j-1}(s,n,m,k)]$$

$$+\lambda E[\phi(X^*(r,n,m,k),X^*(s,n,m,k))]\Big\}, \tag{3.10}$$

where

$$\phi(x,y) = x^i y^{j-1} e^y.$$

Remark 3.1. Putting $m = 0$, $k = 1$ in (3.7), we obtain the recurrence relations for joint moment generating function of order statistics for extended type I generalized logistic distribution in the form

$$\Big(1+\frac{t_2}{\alpha(n-s+1)}\Big)M^{(i,j)}_{X_{n-r+1,n-s+1:n}}(t_1,t_2) = M^{(i,j)}_{X_{n-r+1,n-s+2:n}}(t_1,t_2)$$

$$-\frac{j}{\alpha(n-s+1)}M^{(i,j-1)}_{X_{n-r+1,n-s+1:n}}(t_1,t_2)$$

$$-\frac{\lambda}{\alpha(n-s+1)}\Big\{t_2 M^{(i,j)}_{X_{n-r+1,n-s+1:n}}(t_1,t_2+1)+jM^{(i,j-1)}_{X_{n-r+1,n-s+1:n}}(t_1,t_2+1)\Big\}.$$

That is

$$M^{(i,j)}_{X_{r,s:n}}(t_1,t_2)=\Big(1+\frac{t_1}{\alpha(r-1}\Big)M^{(i,j)}_{X_{r-1,s:n}}(t_1,t_2)+\frac{i}{\alpha(r-1}M^{(i-1,j)}_{X_{r-1,s:n}}(t_1,t_2)$$

$$+\frac{\lambda}{\alpha(r-1)}\Big\{t_1 M^{(i,j)}_{X_{r-1,s:n}}(t_1+1,t_2)+iM^{(i-1,j)}_{X_{r-1,s:n}}(t_1,t_2)\Big\}.$$

Remark 3.2. Substituting $m=-1$ and $k\geq 1$, in Theorem 3.1, we get recurrence relation for joint moment generating function of lower k record values for extended type I generalized logistic distribution in the form

$$\Big(1+\frac{t_2}{\alpha k}\Big)M^{(i,j)}_{X^*(r,n,-1,k),X^*(s,n,-1,k)}(t_1,t_2)=M^{(i,j)}_{X^*(r,n,-1,k),X^*(s-1,n,-1,k)}(t_1,t_2)$$

$$-\frac{j}{\alpha k}M^{(i,j-1)}_{X^*(r,n,-1,k),X^*(s,n,-1,k)}(t_1,t_2)$$

$$-\frac{\lambda}{\alpha k}\Big\{t_2 M^{(i,j)}_{X^*(r,n,-1,k),X^*(s,n,-1,k)}(t_1,t_2+1)$$

$$+jM^{(i,j-1)}_{X^*(r,n,-1,k),X^*(s,n,-1,k)}(t_1,t_2+1)\Big\}.$$

4. Characterization

Let $X^*(r,n,m,k)$, $r=1,2,\ldots,n$ be *lgos* from a continuous population with *df* $F(x)$ and *pdf* $f(x)$, then the conditional *pdf* of $X^*(s,n,m,k)$ given $X^*(r,n,m,k)=x$, $1\leq r<s\leq n$, in view of (1.2) and (1.3), is

$$f_{X^*(s,n,m,k)|X^*(r,n,m,k)}(y|x)=\frac{C_{s-1}}{(s-r-1)!C_{r-1}}[F(x)]^{m-\gamma_r+1}$$

$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y). \tag{4.1}$$

Theorem 4.1. *Let X be an absolutely continuous random variable df $F(x)$ and pdf $f(x)$ on the support $(-\infty,\infty)$ then*

$$E[e^{tX^*(s,n,m,k)}|X^*(l,n,m,k)=x]$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^t (t)_{(p)}}{\lambda^t p!} \Big(\frac{\lambda + e^{-x}}{\lambda}\Big)^p \prod_{j=1}^{s-l} \Big(\frac{\gamma_{l+j}}{\gamma_{l+j} - p/\alpha}\Big), \quad l = r, r+1 \tag{4.2}$$

if and only if

$$F(x) = \Big(\frac{\lambda}{\lambda + e^{-x}}\Big)^{\alpha}, \quad -\infty < x < \infty, \quad \alpha, \lambda > 0.$$

Proof. From (4.1), we have

$$E[e^{tX^*(s,n,m,k)} | X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r-1}}$$

$$\times \int_{-\infty}^{x} e^{ty} \Big[1 - \Big(\frac{F(y)}{F(x)}\Big)^{m+1}\Big]^{s-r-1} \Big(\frac{F(y)}{F(x)}\Big)^{\gamma_s - 1} \frac{f(y)}{F(x)} dy. \tag{4.3}$$

By setting $u = \frac{F(y)}{F(x)} = \Big(\frac{\lambda + e^{-x}}{\lambda + e^{-y}}\Big)^{\alpha}$ from (1.5) in (4.3), we obtain

$$E[e^{tX^*(s,n,m,k)} | X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r-1}}$$

$$\times \int_0^1 [(\lambda + e^{-x}) u^{-1/\alpha} - \lambda]^{-t} u^{\gamma_s - 1} (1 - u^{m+1})^{s-r-1} du$$

$$= \frac{C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r-1}} \sum_{p=0}^{\infty} \frac{(-1)^t (t)_{(p)}}{\lambda^t p!} \Big(\frac{\lambda + e^{-x}}{\lambda}\Big)^p$$

$$\times \int_0^1 u^{\gamma_s - (p/\alpha) - 1} (1 - u^{m+1})^{s-r-1} du. \tag{4.4}$$

Again by setting $z = u^{m+1}$ in (4.4), we get

$$E[e^{tX^*(s,n,m,k)} | X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r}}$$

$$\times \sum_{p=0}^{\infty} \frac{(-1)^t (t)_{(p)}}{\lambda^t p!} \Big(\frac{\lambda + e^{-x}}{\lambda}\Big)^p \int_0^1 z^{\frac{\alpha k - p}{\alpha(m+1)} + n - s - 1} (1 - z)^{s-r-1} dz$$

$$= \frac{C_{s-1}}{C_{r-1} (m+1)^{s-r}} \sum_{p=0}^{\infty} \frac{(-1)^t (t)_{(p)}}{\lambda^t p!} \Big(\frac{\lambda + e^{-x}}{\lambda}\Big)^p \frac{\Gamma\Big(\frac{\alpha k - p}{\alpha(m+1)} + n - s\Big)}{\Gamma\Big(\frac{\alpha k - p}{\alpha(m+1)} + n - r\Big)}$$

$$= \frac{C_{s-1}}{C_{r-1}} \sum_{p=0}^{\infty} \frac{(-1)^t (t)_{(p)}}{\lambda^t p!} \Big(\frac{\lambda + e^{-x}}{\lambda}\Big)^p \frac{1}{\prod_{j=1}^{s-r} (\gamma_{r+j} - p/\alpha)}$$

and hence the result given in (4.2).
To prove sufficient part, we have from (4.1) and (4.2)

$$\frac{C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r-1}} \int_{-\infty}^{\infty} e^{t_2 y} [(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-1}$$
$$\times [F(y)]^{\gamma_s - 1} f(y) dy = [F(x)]^{\gamma_{r+1}} H_r(x), \tag{4.5}$$

where

$$H_r(x) = \sum_{p=0}^{\infty} \frac{(-1)^t (t)_{(p)}}{\lambda^t p!} \Big(\frac{\lambda + e^{-x}}{\lambda}\Big)^p \prod_{j=1}^{s-r} \Big(\frac{\gamma_{r+j}}{\gamma_{r+j} - p/\alpha}\Big).$$

Differentiating (4.5) both sides with respect to x, we get

$$\frac{C_{s-1} [F(x)]^m f(x)}{(s-r-2)! C_{r-1} (m+1)^{s-r-2}} \int_{-\infty}^{\infty} e^{t_2 y} [(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-2}$$
$$\times [F(y)]^{\gamma_s - 1} f(y) dy = H_r'(x) [F(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x) [F(x)]^{\gamma_{r+1} - 1} f(x)$$
$$\gamma_{r+1} H_{r+1}(x) [F(x)]^{\gamma_{r+2} + m} f(x)$$
$$= H_r'(x) [F(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x) [F(x)]^{\gamma_{r+1} - 1} f(x).$$

Therefore,

$$\frac{f(x)}{F(x)} = \frac{H_r'(x)}{\gamma_{r+1} [H_{r+1}(x) - H_r(x)]} = \frac{\alpha}{(1 + \lambda e^x)}$$

which proves that

$$F(x) = \Big(\frac{\lambda}{\lambda + e^{-x}}\Big)^\alpha, \quad -\infty < x < \infty, \quad \alpha, \lambda > 0.$$

Theorem 4.2. *Let X be an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all x, then*

$$M_{X^*(r,n,m,k)}(t) - M_{X^*(r-1,n,m,k)}(t)$$

$$-\frac{t}{\alpha\gamma_r}\Big\{M_{X^*(r,n,m,k)}(t)+\lambda M_{X^*(r-1,n,m,k)}(t+1)\Big\}. \tag{4.6}$$

if and only if

$$F(x)=\Big(\frac{\lambda}{\lambda+e^{-x}}\Big)^{\alpha}, \quad -\infty<x<\infty, \quad \alpha,\lambda>0.$$

Proof. The necessary part follows immediately from equation (2.11). On the other hand if the recurrence relation in equation (4.6) is satisfied, then on using equation (1.2), we have

$$\frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty}e^{tx}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$

$$=\frac{(r-1)C_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}e^{tx}[F(x)]^{\gamma_r+m}f(x)g_m^{r-2}(F(x))dx$$

$$-\frac{tC_{r-1}}{\alpha\gamma_r(r-1)!}\int_{-\infty}^{\infty}e^{tx}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$

$$-\frac{\lambda tC_{r-1}}{\alpha\gamma_r(r-1)!}\int_{-\infty}^{\infty}e^{(t+1)x}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \tag{4.7}$$

Integrating the first integral on the right hand side of equation (4.7), by parts, we get

$$\frac{tC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}e^{tx}[F(x)]^{\gamma_r-1}g_m^{r-1}(F(x))\Big\{F(x)-\frac{1}{\alpha}f(x)-\frac{\lambda e^{x}}{\alpha}f(x)\Big\}dx=0. \tag{4.8}$$

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, [9] to equation (4.8), we get

$$\frac{f(x)}{F(x)}=\frac{\alpha}{(1+\lambda e^{x})}$$

which proves that

$$F(x)=\Big(\frac{\lambda}{\lambda+e^{-x}}\Big)^{\alpha}, \quad -\infty<x<\infty, \quad \alpha,\lambda>0.$$

5. Conclusion

This paper deals with the lower generalized order statistics from the extended type I generalized logistic distribution. Some explicit expressions and recurrence relations for marginal and joint moment generating functions are derived. Characterizing results of the extended type I generalized logistic distribution has been obtained by using conditional expectation and recurrence relation of lower generalized order statistics. Special cases are also deduced.

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Qualitative Theory for Fractional Order Riemann-Liouville Integral Equations in Two Independent Variables *

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Dedicated to the memory of Professor B.G. Pachpatte

Abstract

In this paper, we present some results concerning the existence and uniqueness and global asymptotic stability of solutions for a functional integral equation of Riemann-Liouville fractional order, by using some fixed point theorems for the existence and uniqueness of the solution and by using some techniques of Pachpatte concerning the estimate on the solution.

Keywords and Phrases: *Functional integral equation, Left-sided mixed Riemann-Liouville integral of fractional order, Solution, Estimation, asymptotic stability, fixed point.*

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1. Introduction

Fractional integral equations have recently been applied in various areas of engineering, science, finance, applied mathematics, bio-engineering, radiative transfer, neutron transport and the kinetic theory of gases and others [5, 9, 10, 11, 15, 16]. However, many researchers remain unaware of this field. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas *et al.* [4], Baleanu *et al.* [6], Diethelm [13], Kilbas *et al.* [17], Miller and Ross [18], Podlubny [24], Samko *et al.* [25]. Recently some results on the existence and the attractivity of the solutions of various classes of integral equations have been obtained by Abbas *et al.* [1, 2, 3], Banaś and Zając [7], Darwish *et al.* [12], Pachpatte [19, 20, 21, 22, 23] and the references therein. In most of the above cited papers the main tool was the measure of noncompactness. In [23], Pachpatte proved some results concerning some basic qualitative properties of solutions of the following general partial integral equation of Barbashin type of the form

$$x(t,x) = h(t,x) + \int_0^t f(t,x,s,u(s,x))ds + \int_0^t \int_B g(t,x,s,y,u(s,y))dyds; \quad (1)$$

for $(t,x) \in E$, where $h : \mathbb{R}_+ \times B \to \mathbb{R},\ f : E_1 \times \mathbb{R} \to \mathbb{R},\ g : E^2 \times \mathbb{R} \to \mathbb{R}$ are given functions continuous functions, $\mathbb{R}_+ = [0,+\infty),\ B = \prod_{i=1}^m [a_i, b_i] \subset \mathbb{R}^m (a_i < b_i), E = \mathbb{R}_+ \times B,\ E_1 = \{(t,x,s) : 0 \le s \le t < \infty,\ x \in B\}$. To establish the results, he obtains and uses a variant of a certain integral inequality with explicit estimate.

In this paper, by means of integral inequalities and the fixed point approach, we improve the above results for the following partial integral equation of Riemann-Liouville fractional order of the form

$$u(t,x) = \mu(t,x) + \frac{1}{\Gamma(r_1)} \int_0^t (t-s)^{r_1-1} f(t,x,s,u(s,x))ds$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1} g(t,x,s,y,u(s,y))dyds;\ (t,x) \in J, \quad (2)$$

where $J = \mathbb{R}_+ \times [0,b],\ b > 0,\ r = (r_1, r_2),\ r_1, r_2 \in (0,\infty),\ \mu : J \to \mathbb{R},\ f : J_1 \times \mathbb{R} \to \mathbb{R},\ g : J_2 \times \mathbb{R} \to \mathbb{R}$ are given continuous functions,

$$J_1 = \{(t,x,s) : 0 \le s \le t < \infty,\ x \in [0,b]\},$$

$$J_2 = \{(t, x, s, y) : 0 \leq s \leq t < \infty,\ x \in [0, b],\ y \in [0, b)\},$$

and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt,\ \xi > 0$.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $L^1([0, a] \times [0, b]);\ a, b > 0$ we denote the space of Lebesgue-integrable functions $u : [0, a] \times [0, b] \to \mathbb{R}$ with the norm

$$\|u\|_1 = \int_0^a \int_0^b |u(t, x)|dxdt.$$

As usual, by $C := C(J)$ we denote the space of all continuous functions from J into $\mathbb{R}$. By $BC := BC(J)$ we denote the Banach space of all bounded and continuous functions from J into $\mathbb{R}$ equipped with the standard norm

$$\|u\|_{BC} = \sup_{(t,x)\in J} |u(t, x)|.$$

Definition 1. ([25]) Let $r \in (0, \infty)$. For $u \in L^1([0, b]);\ b > 0$ the expression

$$(I_0^r u)(t) = \frac{1}{\Gamma(r)} \int_0^t (t - s)^{r-1}u(s)ds,$$

is called the left-sided mixed Riemann-Liouville integral of order r.

In particular,

$$(I_0^0 u)(t) = u(t),\ (I_0^1 u)(t) = \int_0^t u(s)ds;\ \text{for almost all } t \in [0, b].$$

For instance, $I_0^r u$ exists for all $r > 0$, when $u \in L^1([0, b])$. Note also that when $u \in C([0, b])$, then $(I_0^r u) \in C([0, b])$,

Example 2.1. *Let $\omega \in (-1, 0) \cup (0, \infty)$ and $r \in (0, \infty)$, then*

$$I_0^r t^\omega = \frac{\Gamma(1 + \omega)}{\Gamma(1 + \omega + r)} t^{\omega+r},\ \textit{for almost all } t \in [0, b].$$

Definition 2. [25] Let $r \in (0,\infty)$ and $u \in L^1([0,a]\times[0,b])$; $a,b>0$. The partial Riemann-Liouville integral of order r of $u(t,x)$ with respect to x is defined by the expression

$$I_{0,t}^r u(t,x) = \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}u(s,x)ds,$$

for almost all $(t,x)\in[0,a]\times[0,b]$.

Analogously, we define the integral

$$I_{0,t}^r u(x,t) = \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}u(x,s)ds,$$

for almost all $(x,t)\in[0,a]\times[0,b]$.

Definition 3. ([26]) Let $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, $\theta=(0,0)$ and $u\in L^1([0,a]\times[0,b])$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}u(s,y)dyds.$$

In particular,

$$(I_\theta^\theta u)(t,x) = u(t,x),\ (I_\theta^\sigma u)(t,x)$$

$$= \int_0^t\int_0^x u(s,y)dyds;\ \text{for almost all } (t,x)\in[0,a]\times[0,b],$$

where $\sigma=(1,1)$.

For instance, $I_\theta^r u$ exists for all $r_1,r_2>0$, when $u\in L^1([0,a]\times[0,b])$. Moreover

$$(I_\theta^r u)(t,0) = (I_\theta^r u)(0,x) = 0;\ t\in[0,a],\ x\in[0,b].$$

Example 2.2. *Let $\lambda,\omega\in(-1,0)\cup(0,\infty)$ and $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, then*

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}t^{\lambda+r_1}x^{\omega+r_2},$$

for almost all $(t,x)\in[0,a]\times[0,b]$.

Let G be an operator from $\Omega \subset BC$; $\Omega \neq \emptyset$ into itself and consider the solutions of equation

$$(Gu)(t,x) = u(t,x). \tag{3}$$

Now we review the concept of attractivity of solutions for equation (1) (see [3]).

Definition 4. Solutions of equation (3) are locally attractive if there exists a ball $B(u_0,\eta)$ in the space BC such that for arbitrary solutions $v = v(t,x)$ and $w = w(t,x)$ of equations (3) belonging to $B(u_0,\eta) \cap \Omega$ we have that for each $x \in [0,b]$

$$\lim_{t\to\infty} (v(t,x) - w(t,x)) = 0. \tag{4}$$

When the limit (4) is uniform with respect to $B(u_0,\eta)\cap\Omega$, solutions of equation (3) are said to be uniformly locally attractive (or equivalently that solutions of (3) are locally asymptotically stable).

Definition 5. The solution $v = v(t,x)$ of equation (3) is said to be globally attractive if (4) hold for each solution $w = w(t,x)$ of (3). If condition (4) is satisfied uniformly with respect to the set Ω, solutions of equation (3) are said to be globally asymptotically stable (or uniformly globally attractive).

Denote by $D_1 := \frac{\partial}{\partial t}$, the partial derivative of a function defined on J_1 (or J_2) with respect to the first variable. In the sequel we will make use of the following Lemma due to Pachpatte.

Lemma 2.3. *([23]) Let $u \in C(J)$, $q, D_1q \in C(J_1), k, D_1k \in C(J_2)$ be positive functions, and $c \geq 0$ is a constant. If*

$$u(t,x) \leq c + \int_0^t q(t,x,s)u(s,x)ds + \int_0^t \int_0^b k(t,x,s,y)u(s,y)dyds;\ (t,x) \in J, \tag{5}$$

then,

$$u(t,x) \leq cP(t,x)\exp\Big(\int_0^t A(\sigma,x)d\sigma\Big);\ (t,x) \in J, \tag{6}$$

where

$$P(t,x) = \exp(Q(t,x)), \tag{7}$$

in which

$$Q(t,x) = \int_0^t \Big[q(\eta,x,\eta) + \int_0^\eta D_1q(\eta,x,\xi)d\xi\Big]d\eta, \tag{8}$$

and

$$A(t,x) = \int_0^b k(t,x,t,y)P(t,y)dy + \int_0^t \int_0^b P(s,y)D_1k(t,x,s,y)dyds; \ (t,x) \in J. \tag{9}$$

3. Main Results

Let us start by defining what we mean by a solution of equation (2).

Definition 6. A function $u \in BC$ is said to be a solution of (2) if u satisfies the equation (2) on J.

Our first result is about the existence and uniqueness of the solution of equation (2).

Theorem 3.1. *Assume that following hypotheses hold*

(H_1) *The function μ is continuous and bounded with*

$$\mu^* = \sup_{(t,x)\in\mathbb{R}_+\times[0,b]} |\mu(t,x)|.$$

(H_2) *There exists a positive function $q \in BC(J_1)$ such that*

$$|f(t,x,s,u) - f(t,x,s,v)| \leq q(t,x,s)|u-v|,$$

$$\textit{for each}(t,x,s) \in J_1 \textit{and } u,v \in \mathbb{R}.$$

Moreover, assume that the function $t \to \int_0^t (t-s)^{r_1-1} f(t,x,s,0)ds$ is bounded on J with

$$f^* = \sup_{(t,x)\in J} \frac{1}{\Gamma(r_1)} \int_0^t (t-s)^{r_1-1}|f(t,x,s,0)|ds.$$

(H_3) *There exists a positive function $k \in BC(J_2)$ such that*

$$|g(t,x,s,y,u) - g(t,x,s,y,v)| \leq k(t,x,s,y)|u-v|,$$

for each $(t,x,s,y) \in J_2$ and $u,v \in \mathbb{R}$. Moreover, assume that the function $t \to \int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1} g(t,x,s,y,0)dyds$ is bounded on J with

$$g^* = \sup_{(t,x)\in J} \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}|g(t,x,s,y,0)|dyds.$$

If

$$q^* + k^* < 1, \tag{10}$$

where

$$q^* = \sup_{(t,x)\in J}\Big[\frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1} q(t,x,s)ds\Big],$$

and

$$k^* = \sup_{(t,x)\in J}\Big[\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1} k(t,x,s,y)dyds\Big],$$

then equation (2) has a unique solution on J.

Proof. Let us define the operator $N : BC \to BC$, such that for each $(t,x) \in J$,

$$(Nu)(t,x) = \mu(t,x) + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1} f(t,x,s,u(s,x))ds$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1} g(t,x,s,y,u(s,y))dyds;\ (t,x)\in J. \tag{11}$$

It is clear that the function $(t,x) \mapsto N(u)(t,x)$ is continuous on J. Now we prove that $N(u) \in BC$ for any $u \in BC$. For arbitrarily fixed $(t,x) \in J$ we

have

$$
\begin{aligned}
|(Nu)(t,x)| &= \Big|\mu(t,x)+\frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}f(t,x,s,u(s,x))ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}g(t,x,s,y,u(s,y))dyds\Big| \\
&\le |\mu(t,x)| + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,u(s,x))-f(t,x,s,0)|ds \\
&+ \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,0)|ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1} \\
&\times |g(t,x,s,y,u(s,y))-g(t,x,s,y,0)|dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}|g(t,x,s,y,0)|dyds \\
&\le |\mu(t,x)| + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}q(t,x,s)|u(s,x)|ds \\
&+ \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,0)|ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)|u(s,y)|dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}|g(t,x,s,y,0)|dyds \\
&\le \mu^* + f^* + g^* + (q^*+k^*)\|u\|_{BC}.
\end{aligned}
$$

Hence $N(u)\in BC$. Let $u,\ v\in BC$. Using the hypotheses, for each $(t,x)\in J$,

we have

$$
\begin{aligned}
&|(Nu)(t,x)-(Nv)(t,x)| \\
&\leq \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,u(s,x))-f(t,x,s,v(s,x))|ds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1} \\
&\times|g(t,x,s,y,u(s,y))-g(t,x,s,y,v(s,y))|dyds \\
&\leq \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}q(t,x,s)|u(s,x)-v(s,x)|ds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)|u(s,y)-v(s,y)|dyds \\
&\leq \sup_{(t,x)\in J}\Big[\frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}q(t,x,s)ds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)dyds\Big]\|u-v\|_{BC} \\
&\leq (q^*+k^*)\|u-v\|_{BC}.
\end{aligned}
$$

From (10), it follows that N has a unique fixed point in BC by Banach contraction principle. The fixed point of N is however a solution of equation (2).

Now, we shall prove the following theorem concerning the estimate on the solution of equation (2).

Theorem 3.2. *Set*

$$d=\mu^*+f^*+g^*. \tag{12}$$

Assume that $(H_1)-(H_3)$ *and the following hypothesis holds*

(H_4) $q_1, D_1q_1\in BC(J_1)$ *and* $k_1, D_1k_1\in BC(J_2)$, *where*

$$q_1(t,x,s)=\frac{1}{\Gamma(r_1)}(t-s)^{r_1-1}q(t,x,s)$$

and

$$k_1(t,x,s,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}(t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y).$$

If u is any solution of (2) on J, then

$$|u(t,x)| \leq dP_1(t,x)\exp\Big(\int_0^t A_1(\sigma,x)d\sigma\Big);\ (t,x)\in J, \tag{13}$$

where

$$P_1(t,x) \leq \exp(Q_1(t,x)), \tag{14}$$

in which

$$Q_1(t,x) \leq \int_0^t \Big[q_1(\eta,x,\eta)+\int_0^\eta D_1q_1(\eta,x,\xi)d\xi\Big]d\eta, \tag{15}$$

and

$$A_1(t,x) \leq \int_0^b k_1(t,x,t,y)P_1(t,y)dy+\int_0^t\int_0^b P_1(s,y)D_1k_1(t,x,s,y)dyds. \tag{16}$$

Proof. Using the fact that u is a solution of (2) and hypotheses, then for each $(t,x)\in J$, we have

$$\begin{aligned}
|u(t,x)| &\leq |\mu(t,x)|+\frac{1}{\Gamma(r_1)}\int_0^t(t-s)^{r_1-1}|f(t,x,s,u(s,x))-f(t,x,s,0)|ds\\
&+\frac{1}{\Gamma(r_1)}\int_0^t(t-s)^{r_1-1}|f(t,x,s,0)|ds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b(t-s)^{r_1-1}(b-y)^{r_2-1}\\
&\times|g(t,x,s,y,u(s,y))-g(t,x,s,y,0)|dyds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b(t-s)^{r_1-1}(b-y)^{r_2-1}|g(t,x,s,y,0)|dyds\\
&\leq d+\frac{1}{\Gamma(r_1)}\int_0^t(t-s)^{r_1-1}|f(t,x,s,u(s,x))-f(t,x,s,0)|ds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b(t-s)^{r_1-1}(b-y)^{r_2-1}\\
&\times|g(t,x,s,y,u(s,y))-g(t,x,s,y,0)|dyds\\
&\leq d+\frac{1}{\Gamma(r_1)}\int_0^t(t-s)^{r_1-1}q(t,x,s)|u(x,s)|ds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b(t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)|u(s,y)|dyds.
\end{aligned}\tag{17}$$

Now an application of Lemma 2.3, to (17) yields (13).

Theorem 3.3. *Set*

$$\overline{d} := f^* + g^* + \mu^*(q^* + k^*). \tag{18}$$

Assume that $(H_1) - (H_3)$ hold. If u is any solution of (2) on J, then

$$|u(t,x) - \mu(t,x)| \leq \overline{d}P_1(t,x)\exp\Big(\int_0^t A_1(\sigma,x)d\sigma\Big); \ (t,x) \in J, \tag{19}$$

where P_1 and A_1 are given by (14) and (16), respectively.

Proof. Let $h(t,x) = |u(t,x) - \mu(t,x)|$. Using the fact that u is a solution of (2) and from the hypotheses, for each $(t,x) \in J$, we have

$$\begin{aligned}
h(t,x) &\leq \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,u(s,x)) - f(t,x,s,\mu(s,x))|ds \\
&+ \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,\mu(s,x))|ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1} \\
&\times |g(t,x,s,y,u(s,y)) - g(t,x,s,y,\mu(s,x))|dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}|g(t,x,s,y,\mu(s,x))|dyds \\
&\leq \overline{d} + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,u(s,x)) - f(t,x,s,\mu(s,x))|ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1} \\
&\times |g(t,x,s,y,u(s,y)) - g(t,x,s,y,\mu(s,x))|dyds \\
&\leq \overline{d} + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}q(t,x,s)h(x,s)ds
\end{aligned}$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)h(s,y)dyds. \tag{20}$$

Now an application of Lemma 2.3, to (20) yields (19).

We next prove under more appropriate conditions on the functions involved in (2) that the solutions tends exponentially toward zero as $t \to \infty$.

Theorem 3.4. *Assume that the following hypotheses hold*

(H_5) *There exist constants $\alpha > 0$ and $M \geq 0$ such that*

$$|\mu(t,x)| \leq Me^{-\alpha t}; \tag{21}$$

$$|f(t,x,s,u) - f(t,x,s,v)| \leq q(t,x,s)e^{-\alpha(t-s)}|u-v|; \tag{22}$$

$$|g(t,x,s,y,u) - g(t,x,s,y,v)| \leq k(t,x,s,y)e^{-\alpha(t-s)}|u-v|; \tag{23}$$

and $f(t,x,s,0) = g(t,s,s,y,0) = 0$; for each $(t,x) \in J$, $(t,x,s) \in J_1$, $(t,x,s,y) \in J_2$, $u,v \in \mathbb{R}$, and the functions q, k be as in Theorem 3.1,

(H_6) $\sup_{(t,x)\in J} Q_1(t,x) < \infty$, $\int_0^\infty A_1(\sigma,x)d\sigma < \infty$, *where Q_1 and A_1 are given by (15) and (16).*

If u is any solution of (2) on J, then all solutions of equation (2) are uniformly globally attractive on J.

Proof. From the hypotheses, for each $(t,x) \in J$, we have that

$$\begin{aligned}
|u(t,x)| &\leq |\mu(t,x)| + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,u(s,x)) - f(t,x,s,0)|ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1} \\
&\times |g(t,x,s,y,u(s,y)) - g(t,x,s,y,0)|dyds \\
&\leq Me^{-\alpha t} + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}q(t,x,s)e^{-\alpha(t-s)}|u(x,s)|ds
\end{aligned}$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)e^{-\alpha(t-s)}|u(s,y)|dyds. \tag{24}$$

From (24), we get

$$|u(t,x)|e^{\alpha t} \leq M + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}q(t,x,s)e^{\alpha s}|u(x,s)|ds$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)e^{\alpha s}|u(s,y)|dyds. \tag{25}$$

Now an application of Lemma 2.3 to (25) yields

$$|u(t,x)|e^{\alpha t} \le MP_1(t,x)\exp\Big(\int_0^t A_1(\sigma,x)d\sigma\Big);\ (t,x)\in J. \tag{26}$$

Multiplying both sides of (26) by $e^{-\alpha t}$ and in view of (H_6), we get

$$\lim_{t\to\infty}|u(t,x)| \le \lim_{t\to\infty} MP_1(t,x)\exp\Big(-\alpha t+\int_0^t A_1(\sigma,x)d\sigma\Big)=0.$$

Hence, the solution u tends to zero as $t\to\infty$. Consequently, all solutions of equation (2) are uniformly globally attractive on J.

4. An Example

To illustrate our results, we consider the following partial integral equation of Riemann-Liouville fractional order of the form

$$u(t,x)=\frac{e^{x-t}}{1+t+x^2}+\frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}f(t,x,s,u(s,x))ds$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^1 (t-s)^{r_1-1}(1-y)^{r_2-1}g(t,x,s,y,u(s,y))dyds;\ (t,x)\in\mathbb{R}_+\times[0,1], \tag{27}$$

where $r_1,r_2\in(0,\infty)$,

$$\begin{cases} f(t,x,s,u)=\dfrac{x^2t^{-r_1}s^{-\frac12}\sin s\sin t}{2c(1+t^{-\frac12})(1+|u|)}; for (t,x,s)\in J_1,\ st\ne 0 and u\in\mathbb{R},\\ f(t,x,0,u)=f(0,x,0,u)=0,\end{cases}$$

$$J_1=\{(t,x,s): 0\le s\le t<\infty,\ x\in[0,1]\},$$

$$c:=\frac{\Gamma(\frac12)}{\Gamma(\frac12+r_1)}+\frac{\Gamma(\frac12)e}{\Gamma(\frac12+r_1)\Gamma(1+r_2)},$$

$$\begin{cases} g(t,x,s,y,u)=\dfrac{t^{-r_1}s^{-\frac12}e^{x-y-\frac1s-\frac1t}}{2c(1+t^{-\frac12})(1+|u|)}; for (t,x,s,y)\in J_2,\ st\ne 0 and\ u\in\mathbb{R},\\ g(t,x,0,y,u)=g(0,x,0,y,u)=0,\end{cases}$$

and

$$J_2 = \{(t,x,s,y) : 0 \le s \le t < \infty,\ x \in [0,1],\ y \in [0,1)\}.$$

Set

$$\mu(t,x) = \frac{e^{x-t}}{1+t+x^2};\ (t,x) \in J.$$

We can see that the function μ is continuous and bounded with $\mu^* = e$. For each $u,v \in \mathbb{R}$ and $(t,x,s) \in J_1$, we have

$$|f(t,x,s,u) - f(t,x,s,v)| \le \frac{1}{2c(1+t^{-\frac{1}{2}})}\left(x^2 t^{-r_1} s^{-\frac{1}{2}} |\sin s \sin t|\right)|u-v|,$$

and for each $u,v \in \mathbb{R}$ and $(t,x,s,y) \in J_2$, we have

$$|g(t,x,s,y,u) - g(t,x,s,y,v)| \le \frac{1}{2c(1+t^{-\frac{1}{2}})}\left(t^{-r_1} s^{-\frac{1}{2}} e^{x-y-t-\frac{1}{s}-\frac{1}{t}}\right)|u-v|.$$

Hence condition (H_2) is satisfied with

$$\begin{cases} q(t,x,s) = \dfrac{1}{2c(1+t^{-\frac{1}{2}})}\left(x^2 t^{-r_1} s^{-\frac{1}{2}} |\sin s \sin t|\right);\ st \neq 0, \\ \qquad q(t,x,0) = q(0,x,0) = 0, \end{cases}$$

and condition (H_3) is satisfied with

$$\begin{cases} k(t,x,s,y) = \dfrac{1}{2c(1+t^{-\frac{1}{2}})}\left(t^{-r_1} s^{-\frac{1}{2}} e^{x-y-t-\frac{1}{s}-\frac{1}{t}}\right);\ st \neq 0, \\ \qquad k(t,x,0,y) = k(0,x,0,y) = 0. \end{cases}$$

We shall show that condition (10) holds with $b = 1$. Indeed

$$\begin{aligned} & \frac{1}{\Gamma(r_1)} \int_0^t (t-s)^{r_1-1} q(t,x,s)ds \\ \le\ & \frac{1}{2c(1+t^{-\frac{1}{2}})\Gamma(r_1)} \int_0^t (t-s)^{r_1-1} x^2 t^{-r_1} s^{-\frac{1}{2}} ds \\ =\ & x^2 t^{-r_1} t^{-\frac{1}{2}+r_1} \frac{\Gamma(\frac{1}{2})}{2c(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{2}+r_1)} \\ \le\ & \frac{\Gamma(\frac{1}{2})}{2c(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{2}+r_1)} t^{-\frac{1}{2}}, \end{aligned}$$

then

$$q^* \leq \frac{\Gamma(\frac{1}{2})}{2c\Gamma(\frac{1}{2}+r_1)}.$$

Also,

$$\begin{aligned}
& \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)dyds \\
\leq\ & \frac{1}{2c(1+t^{-\frac{1}{2}})\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^1 (t-s)^{r_1-1}(1-y)^{r_2-1}t^{-r_1}s^{-\frac{1}{2}}e^x dyds \\
\leq\ & e^x t^{-r_1}t^{-\frac{1}{2}+r_1}\frac{\Gamma(\frac{1}{2})}{2c(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)} \\
\leq\ & \frac{\Gamma(\frac{1}{2})et^{-\frac{1}{2}}}{2c(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)},
\end{aligned}$$

then

$$k^* \leq \frac{e\Gamma(\frac{1}{2})}{2c\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)}.$$

Thus,

$$q^*+k^* \leq \frac{1}{2c}\left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+r_1)}+\frac{\Gamma(\frac{1}{2})e}{\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)}\right)=\frac{1}{2}<1,$$

which is satisfied for each $r_1, r_2 \in (0,\infty)$. Consequently Theorem 3.1 implies that equation (27) has a unique solution defined on $\mathbb{R}_+ \times [0,1]$.

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Sufficient Conditions for Hypergeometric Functions to be in A Certain Class of Holomorphic Functions *

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Abstract

In the present investigation our main objective is to find coefficient estimates, sufficient condition for the function $f(z) \in \mathcal{A}$ to belong to the class $R_{\gamma}^{\tau}(A,B)$ and finding connections between the classes $R_{\gamma}^{\tau}(A,B)$ and $k-\mathcal{UCV}$ by making use of the Hohlov operator [5] .

Keywords and Phrases: *Analytic functions, Subordination, Schwarz functions, Gaussian Hypergeometric functions, $k-\mathcal{UCV}$ functions, Starlike functions, Convex functions, Univalent functions.*

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ that are univalent in $\mathbb{U}$. A function $f(z)$ in $\mathcal{A}$ is said

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to be in class $\mathcal{S}^*$ of starlike functions of order zero in $\mathbb{U}$, if $\Re\left(\frac{zf'(z)}{f(z)}\right) > 0$ for $z \in \mathbb{U}$. Let $\mathcal{K}$ denote the class of all functions $f \in \mathcal{A}$ that are convex. Also, f is convex if and only if $zf'(z)$ is starlike. A function $f \in \mathcal{A}$ is said to be close-to-convex of order α $(0 \leq \alpha < 1)$ with respect to a fixed starlike function $g \in \mathcal{S}^*$ if and only if $\Re\left(\frac{zf'(z)}{g'(z)}\right) > \alpha$ for $z \in \mathbb{U}$. For more details about these classes see [3]. Furthermore, $f \in \mathcal{A}$, then $f \in k - \mathcal{UCV}$ iff

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathbb{U},\ 0 \leqq k < \infty). \tag{1.2}$$

The class $k - \mathcal{UCV}$ was introduced by Kanas and Wisniowska [6], where its geometric definition and connection with the conic domains were considered. In particular $0 - \mathcal{UCV} = \mathcal{K}$.

If $f, g \in \mathcal{H}$,where $\mathcal{H}$ denote the class of holomorphic functions on unit disk $\mathbb{U}$, then the function f is said to be subordinate to g, written as $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function $w \in \mathcal{H}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(w(z))$.
In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

The Gaussian hypergeometric function defined by the series

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \tag{1.3}$$

is analytic in the unit disk $\mathbb{U}$. It arises naturally in the study of second order linear differential equations with regular singular points. In (1.3), $(a)_0 = 1$ for $a \neq 0$ and for each positive integer n, $(a)_n = a(a+1)...(a+n-1)$ is the Pochhammer symbol. To avoid division by 0, the parameter c in (1.3) should be neither zero nor a negative integer. If a or b is 0 or a negative integer, then the power series reduces to a polynomial . Results regarding ${}_2F_1(a, b; c; z)$ when $\Re(c - a - b)$ is positive, zero or negative are abundant in the literature. In particular when $\Re(c - a - b) > 0$, the function ${}_2F_1(a, b; c; z)$ is bounded. This and the zero balanced case $\Re(c - a - b) = 0$ are discussed in detail by many authors (see [9, 13]). For interesting results regarding $\Re(c - a - b) < 0$, see [14].

The hypergeometric function ${}_2F_1(a, b; c; z)$ has been extensively studied by

various authors and play an important role in Geometric Function Theory. It is useful in unifying various functions by giving appropriate values to the parameters $a, b.$ and c. We refer to [2, 4, 10, 13] and reference therein for some important results.

The normalized hypergeometric function $z{}_2F_1(a,b;c;z)$ has a series expansion of the form

$$z{}_2F_1(a,b;c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n. \tag{1.4}$$

Consider the convolution operator by taking the convolution between functions $f(z)$ of the form (1.1) and a normalized hypergeometric functions of the form $z{}_2F_1(a,b;c;z)$:

$$H_{a,b,c}(f)(z) = z{}_2F_1(a,b;c;z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n, \tag{1.5}$$

which was investigated by Hohlov [5]. This three-parameter family of operators given by (1.5) contains most of the known linear integral or differential operators as special cases. In particular , if $a = 1$ in (1.5), then $H_{1,b,c}$ is the operator $\mathcal{L}(b,c)$ due to Carlson and Shaffer [2] which was defined by

$$\mathcal{L}(b,c)f(z) = z{}_2F_1(1,b;c;z) * f(z). \tag{1.6}$$

Note that $z{}_2F_1(1,b;c;z) = \phi(b;c;z)$ is known as incomplete beta function.

In particular, the restriction $b = 1+\delta, c = 2+\delta$ with $\Re\,\delta > -1$ on the operator $\mathcal{L}(b,c)f(z)$ gives the Bernardi operator

$$\mathcal{B}_\delta(f)(z) = \mathcal{L}(\delta+1,\delta+2)(f)(z) = (1+\delta)\int_0^1 t^{\delta-1} f(tz)dt, \tag{1.7}$$

which reduces to the Alexander and Libera transforms, respectively, for $\delta = 1$ and $\delta = 2$. It is interesting to note that these operators are all example of the zero-balanced case $\Re(c-a-b) = 0$ in $H_{1,b,c}(f)(z)$.

Throughout this work, we frequently use the well-known formula

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)},\ (\Re(c-a-b) > 0, c \in \mathbb{C}\backslash\mathbb{Z}_0^-). \tag{1.8}$$

Motivated by the class introduced by Swaminathan [16], Bansal [1] introduced the class $R_\gamma^\tau(A,B)$ as follows:
Definition 1.1. Let $0 \leqq \gamma \leqq 1$, $\tau \in \mathbb{C}\setminus\{0\}$. A function $f \in \mathcal{A}$ is in the class $R_\gamma^\tau(A,B)$, if

$$1+\frac{1}{\tau}(f'(z)+\gamma z f''(z)-1) \prec \frac{1+Az}{1+Bz}\ (-1 \leqq B < A \leqq 1; z \in \mathbb{U}), \qquad (1.9)$$

which is equivalent to saying that

$$\left|\frac{f'(z)+\gamma z f''(z)-1}{\tau(A-B)-B(f'(z)+\gamma z f''(z)-1)}\right| < 1. \qquad (1.10)$$

We list few particular cases of this class discussed in the literature
[1] $R_\gamma^\tau(1-2\beta,-1)=R_\gamma^\tau(\beta)$ for $0 \leqq \beta < 1$, $\tau = \mathbb{C}\setminus\{0\}$ was discussed recently by Swaminathan [16].
[2] The class $R_\gamma^\tau(1-2\beta,-1)$ for $\tau = e^{i\eta}cos\eta$ where $-\pi/2 < \eta < \pi/2$ is considered in [11] (see also [12]).
[3] The class $R_1^\tau(0,-1)$ with $\tau = e^{i\eta}cos\eta$ was considered in [7] with reference to the univalency of partial sums.
[4] $f \in R_\gamma^{e^{i\eta}cos\eta}(1-2\beta,-1)$ whenever $zf'(z) \in P_\gamma^\tau(\beta)$, the class considered in [17].
For geometric aspects of these classes see the corresponding references.

Our main objective in the present paper is to find coefficient estimates, sufficient condition for the functions of the form (1.1) to belong to the class $R_\gamma^\tau(A,B)$ and finding connections between the classes $R_\gamma^\tau(A,B)$ and $k-\mathcal{UCV}$ by making use of the Hohlov operator defined by (1.5). Each of the following lemmas will be required in our investigation.
Lemma A. (See [15]). *Let*

$$h(z) = 1+\sum_{n=1}^{\infty} c_n z^n \prec 1+\sum_{n=1}^{\infty} C_n z^n = H(z)\ (z \in \mathbb{U}). \qquad (1.11)$$

If the function H is univalent in $\mathbb{U}$ and $H(\mathbb{U})$ is a convex set, then

$$|c_n| \leqq |C_1|. \qquad (1.12)$$

Lemma B. (See [6]). *Let $f \in \mathcal{A}$ be of the form* (1.1)*. If for some $k (0 \leqq k < \infty)$, the following inequality:*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leqq \frac{1}{k+2} \qquad (1.13)$$

holds true, then $f \in k-\mathcal{UCV}$. *The number* $\frac{1}{k+2}$ *cannot be increased.*

Lemma C. (See [8]). *Let* $-1 \leqq B_2 \leqq B_1 < A_1 \leqq A_2 \leqq 1$, *then*

$$\frac{1+A_1z}{1+B_1z} \prec \frac{1+A_2z}{1+B_2z}\ (z \in \mathbb{U}).$$

2. Main Results

We first give the following result related to the coefficient of $f(z) \in R^{\tau}_{\gamma}(A,B)$.
Theorem 2.1. *Let* $f(z) \in \mathcal{A}$ *is of the form* (1.1). *If* $f(z)$ *is in* $R^{\tau}_{\gamma}(A,B)$, *then*

$$|a_n| \leqq \frac{|\tau|\,(A-B)}{n\,[1+\gamma(n-1)]}\ \ (n \in \mathbb{N}\backslash\{1\})\,. \tag{2.1}$$

Proof. If $f(z)$ of the form (1.1) belongs to in $R^{\tau}_{\gamma}(A,B)$, then by definition

$$1+\frac{1}{\tau}(f'(z)+\gamma zf''(z)-1) \prec \frac{1+Az}{1+Bz} = h(z)\ (-1 \leqq B < A \leqq 1; z \in \mathbb{U}), \tag{2.2}$$

where $h(z)$ is obviously convex univalent in $\mathbb{U}$ under the stated conditions on A and B. Using (1.1) and doing Binomial expansion of $(1+Bz)^{-1}$ in (2.2), we have

$$1+\frac{1}{\tau}(f'(z)+\gamma zf''(z)-1)$$

$$= 1+\sum_{n=1}^{\infty} \tfrac{(n+1)(1+n\gamma)}{\tau}\ a_{n+1}z^n \prec 1+(A-B)z-B(A-B)z^2+...(z \in \mathbb{U}).$$

Now, by applying Lemma A we get the desired result. □

It is easy to find the sufficient condition for $f(z)$ to be in $R^{\tau}_{\gamma}(A,B)$ under standard techniques. Hence we state the following result without proof.
Theorem 2.2. *Let* $f(z) \in \mathcal{A}$. *Then a sufficient condition for* $f(z)$ *to be in* $R^{\tau}_{\gamma}(A,B)$ *is*

$$\sum_{n=2}^{\infty} n\,[1+\gamma(n-1)]\,|a_n| \leqq \frac{|\tau|\,(A-B)}{|B|+1}. \tag{2.3}$$

The result is sharp for the function

$$f(z) = z+\frac{|\tau|\,(A-B)}{n\,[1+\gamma(n-1)]\,(1+|B|)}z^n\ (n \in \mathbb{N}\backslash\{1\}) \tag{2.4}$$

Remark 2.1. For $B = -1$ and $A = 1 - 2\beta$ $(0 \leqq \beta < 1)$ Theorem 2.1 and 2.2 gives corresponding result of [16].

Theorem 2.3. *Let* $-1 \leqq B_2 \leqq B_1 < A_1 \leqq A_2 \leqq 1$. *Then*

$$R_\gamma^\tau(A_1, B_1) \subset R_\gamma^\tau(A_2, B_2). \tag{2.5}$$

Proof. Let $f \in R_\gamma^\tau(A_1, B_1)$ then by Definition 1.1 of the class $f \in R_\gamma^\tau(A_1, B_1)$ we have

$$1 + \frac{1}{\tau}(f'(z) + \gamma z f''(z) - 1) \prec \frac{1 + A_1 z}{1 + B_1 z}.$$

Since $-1 \leqq B_2 \leqq B_1 < A_1 \leqq A_2 \leqq 1$, by Lemma C, we have

$$1 + \frac{1}{\tau}(f'(z) + \gamma z f''(z) - 1) \prec \frac{1 + A_1 z}{1 + B_1 z} \prec \frac{1 + A_2 z}{1 + B_2 z}.$$

Which implies that $R_\gamma^\tau(A_1, B_1) \subset R_\gamma^\tau(A_2, B_2)$. $\square$

Theorem 2.4. *Suppose that* $a,\ b \in \mathbb{C}\backslash\{0\}$ *and* $\Re(c) > |a|+|b|$. *If* $f \in R_\gamma^\tau(A, B)$ *and the inequality*

$$_2F_1\left(|a|, |b|; \Re(c); 1\right) \leqq \frac{|B| + 2}{|B| + 1} \tag{2.6}$$

holds true, then $z\ {}_2F_1(a, b; c; z) * f(z) \in R_\gamma^\tau(A, B)$.

Proof. Using Theorem 2.2 and (1.5) it is sufficient to prove that

$$\sum_{n=2}^{\infty} n[1 + \gamma(n-1)] \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| |a_n| \leqq \frac{|\tau|(A - B)}{1 + |B|}.$$

Applying Theorem 2.1, for $f \in R_\gamma^\tau(A, B)$, we have

$$\sum_{n=2}^{\infty} n[1 + \gamma(n-1)] \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| |a_n| \leqq |\tau|(A - B) \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(\Re(c))_{n-1}(1)_{n-1}}$$

$$= |\tau|(A - B) \left[{}_2F_1\left(|a|, |b|; \Re(c); 1\right) - 1\right] \leqq \frac{|\tau|(A - B)}{|B| + 1} \text{ (In view of(2.6))}.\square$$

If we set $\gamma = 0, B = -1, A = 1 - 2\alpha$ $(0 \leqq \alpha < 1)$ and $\tau = 1$, we get the functions in the class $R_\gamma^\tau(A, B)$ satisfying the analytic criterion $\Re(f') > \alpha$ which implies that $f(z)$ is close-to-convex of order α with respect to the starlike function $g(z) = z$. Hence the following result is immediate:

Corollary 2.1. *Suppose that $a,\ b \in \mathbb{C}\backslash\{0\}$ and $\Re(c) > |a| + |b|$. If $f \in \mathcal{A}$ of form* (1.1) *satisfying $\Re(f') > \alpha$, and the inequality*

$$\frac{\Gamma(\Re(c) - |a| - |b|)\Gamma(\Re(c))}{\Gamma(\Re(c) - a)\Gamma(\Re(c) - b)} \leqq \frac{3}{2} \tag{2.7}$$

holds true,then $z\ {}_2F_1(a, b; c; z) * f(z)$ is close-to-convex of order α with respect to the starlike function $g(z) = z$.

Theorem 2.5. *Let a, b, c and γ satisfy the hypergeometric inequality*

$${}_2F_1(|a|, |b|; \Re(c); 1)\left[1 + \frac{(1+2\gamma)|ab|}{\Re(c) - |a| - |b| - 1} + \frac{\gamma(|a|)_2(|b|)_2}{(\Re(c) - |a| - |b| - 1)(\Re(c) - |a| - |b| - 2)}\right] - 1$$

$$\leqq \frac{|\tau|\,(A - B)}{|B| + 1}, \tag{2.8}$$

with $a,\ b \in \mathbb{C}\backslash\{0\}$ and $\Re(c) - |a| - |b| - 2 > 0$. Then $z\ {}_2F_1(a, b; c; z)$ is in $R^{\tau}_{\gamma}(A, B)$.

Proof. Using Theorem 2.2 it is sufficient to prove that

$$\sum_{n=2}^{\infty} n\,[1 + \gamma(n-1)] \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \leqq \frac{|\tau|\,(A - B)}{|B| + 1}.$$

It is easy to see that, the left hand side of the above inequality is

$$S = \sum_{n=2}^{\infty} n\,[1 + \gamma(n-1)] \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right|$$

$$= \sum_{n=2}^{\infty} [1 + (1+2\gamma)(n-1) + \gamma(n-1)(n-2)] \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right|$$

$$\leqq \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(\Re(c))_{n-1}(1)_{n-1}} + (1+2\gamma)\frac{|ab|}{\Re(c)} \sum_{n=2}^{\infty} \frac{(|a|+1)_{n-2}(|b|+1)_{n-2}}{(\Re(c)+1)_{n-2}(1)_{n-2}}$$

$$+\gamma \frac{(|a|)_2(|b|)_2}{(\Re(c))_2} \sum_{n=3}^{\infty} \frac{(|a|+2)_{n-3}(|b|+2)_{n-3}}{(\Re(c)+2)_{n-3}(1)_{n-3}}$$

$$= {}_2F_1(|a|, |b|; \Re(c); 1)\left[1 + \frac{(1+2\gamma)|ab|}{\Re(c) - |a| - |b| - 1} + \frac{\gamma(|a|)_2(|b|)_2}{(\Re(c) - |a| - |b| - 1)(\Re(c) - |a| - |b| - 2)}\right] - 1$$

$$\leqq \frac{|\tau|\,(A - B)}{|B| + 1} \text{ (In view of(2.8)).}\ \Box$$

If we set $\gamma = 0, A = 1 - 2\alpha$ $(0 \leqq \alpha < 1), B = -1$ and $\tau = 1$ in Theorem 2.5, we get the following result:

Corollary 2.2. *Let a, b and c satisfy the hypergeometric inequality*

$$ {}_2F_1(|a|, |b|; \Re(c); 1)\left[1 + \frac{|ab|}{\Re(c) - |a| - |b| - 1}\right] \leqq 2 - \alpha, \tag{2.9}$$

with $a,\ b \in \mathbb{C}\backslash\{0\}$ and $\Re(c - |a| - |b| - 1) > 0$, then, $z\ {}_2F_1(a, b; c; z)$ is close-to-convex of order α with respect to the starlike function $g(z) = z$. □

Theorem 2.6. *Suppose that $a,\ b \in \mathbb{C}\backslash\{0\},\ |a| \neq 1, |b| \neq 1, \Re(c) \neq 1$ and $\Re(c) > |a| + |b|$.*
If $f \in R_1^{\tau}(A, B)$ and, for some k $(0 \leqq k < \infty)$, the inequality

$$ {}_2F_1(|a|, |b|; \Re(c); 1) - \frac{\Re(c) - 1}{(|a| - 1)(|b| - 1)}\left({}_2F_1(|a| - 1, |b| - 1; \Re(c) - 1; 1) - 1\right) \leqq \frac{1}{|\tau|\,(A - B)\,(k + 2)} \tag{2.10}$$

holds true, then $z{}_2F_1(a, b; c; z) * f(z) \in k - \mathcal{UCV}$.

Proof. For $f \in R_1^{\tau}(A, B)$ of form (1.1), by applying Theorem 2.1, we have

$$\sum_{n=2}^{\infty} n(n-1)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right||a_n| \leqq \sum_{n=2}^{\infty}\frac{n(n-1)|\tau|(A-B)}{n^2}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(\Re(c))_{n-1}(1)_{n-1}}$$

$$= |\tau|(A - B)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(\mathrm{Re}(c))_{n-1}(1)_{n-1}} - |\tau|(A - B)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(\mathrm{Re}(c))_{n-1}(1)_{n}}$$

$$= |\tau|(A - B)\left[{}_2F_1(|a|, |b|; \Re(c); 1) - 1\right] - |\tau|(A - B)\frac{\Re(c) - 1}{(|a| - 1)(|b| - 1)}\sum_{n=2}^{\infty}\frac{(|a| - 1)_n(|b| - 1)_n}{(\Re(c) - 1)_n(1)_n}$$

$$= |\tau|(A - B)\left[{}_2F_1(|a|, |b|; \Re(c); 1) - \frac{\Re(c) - 1}{(|a| - 1)(|b| - 1)}\left({}_2F_1(|a| - 1, |b| - 1; \Re(c) - 1; 1) - 1\right)\right].$$

Finally, if we make use of (2.10) in above, we find that

$$\sum_{n=2}^{\infty} n(n-1)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right||a_n| \leqq \frac{1}{k+2}\ (0 \leqq k < \infty),$$

which, in view of (1.5) and Lemma B, immediately proves the inclusion property asserted by Theorem 2.6.□

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Applications of Differential Subordination for Functions with Fixed Second Coefficient to Geometric Function Theory*

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Abstract

The theory of second order differential subordination of S. S. Miller and P. T. Mocanu [Differential Subordinations, Dekker, New York, 2000] was recently extended to functions with fixed initial coefficient

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by R. M. Ali, S. Nagpal and V. Ravichandran [Second-order differential subordination for analytic functions with fixed initial coefficient, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), 611–629] and applied to obtain several generalization of classical results in geometric function theory. In this paper, further applications of this subordination theory is given. In particular, several sufficient conditions related to starlikeness, convexity, close-to-convexity of normalized analytic functions are derived.

Keywords and Phrases: *Analytic functions, Starlike functions, Convex functions, Subordination, Fixed second coefficient.*

1. Introduction

For univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the famous Bieberbach theorem shows that $|a_2| \leq 2$ and this bound for the second coefficient yields the growth and distortion bounds as well as covering theorem. In view of the influence of the second coefficient in the properties of univalent functions, several authors have investigated functions with fixed second coefficient. For a brief survey of the various developments, mainly on radius problems, from 1920 to this date, see the recent work by Ali *et al.* [2]. The theory of first-order differential subordination was developed by Miller and Mocanu and a very comprehensive account of the theory and numerous applications can be found in their monograph [9]. Ali *et al.* [4] have extended this well-known theory of differential subordination to the functions with preassigned second coefficient. Nagpal and Ravichandran [10] have applied the results in [4] to obtain several extensions of well-known results to the functions with fixed second coefficient. In this paper, we continue their investigation by deriving several sufficient conditions for starlikeness of functions with fixed second coefficient.

For convenience, let $\mathcal{A}_{n,b}$ denote the class of all functions $f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \cdots$ where $n \in \mathbb{N} = \{1, 2, \dots\}$ and b is a fixed non-negative real number. For fixed $\mu \geq 0$, and $n \in \mathbb{N}$, let $\mathcal{H}_{\mu,n}$ consists of analytic functions p on $\mathbb{D}$ of the form

$$p(z) = 1 + \mu z^n + p_{n+1}z^{n+1} + \cdots \quad (z \in \mathbb{D}). \tag{1.1}$$

Let Ω be a subset of $\mathbb{C}$ and the class $\Psi_{\mu,n}[\Omega]$ consists of those functions $\psi : \mathbb{C}^2 \to \mathbb{C}$ that are continuous in a domain $D \subset \mathbb{C}^2$ with $(1, 0) \in D$, $\psi(1, 0) \in \Omega$,

and satisfy the admissibility condition: $\psi(i\rho, \sigma) \notin \Omega$ whenever $(i\rho, \sigma) \in D$, $\rho \in \mathbb{R}$, and

$$\sigma \leq -\frac{1}{2}\left(n + \frac{2-\mu}{2+\mu}\right)(1+\rho^2). \tag{1.2}$$

When $\Omega = \{w : \operatorname{Re} w > 0\}$, let $\Psi_{\mu,n} := \Psi_{\mu,n}[\Omega]$. The following theorem is needed to prove our main results.

Theorem 1.1. [4, Theorem 3.4] *Let $p \in \mathcal{H}_{\mu,n}$ with $0 < \mu \leq 2$. Let $\psi \in \Psi_{n,\mu}$ with associated domain D. If $(p(z), zp'(z)) \in D$ and $\operatorname{Re}\psi(p(z), zp'(z)) > 0$, then $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$.*

For $\alpha \neq 1$, let

$$\mathcal{S}^*(\alpha) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \right\}.$$

The function $p_\alpha(z) := (1+(1-2\alpha)z)/(1-z)$ maps $\mathbb{D}$ onto $\{w \in \mathbb{C} : \operatorname{Re} w > \alpha\}$ for $\alpha < 1$ and onto $\{w \in \mathbb{C} : \operatorname{Re} w < \alpha\}$ for $\alpha > 1$. Therefore, for $\alpha < 1$, $\mathcal{S}^*(\alpha)$ is the class of starlike functions of order α consisting of functions $f \in \mathcal{A}$ for which $\operatorname{Re}(zf'(z)/f(z)) > \alpha$. For $\alpha > 1$, $\mathcal{S}^*(\alpha)$ reduces to the class $\mathcal{M}(\alpha)$ consisting of $f \in \mathcal{A}$ satisfying $\operatorname{Re}(zf'(z)/f(z)) < \alpha$. The latter class $\mathcal{M}(\alpha)$ and its subclasses were investigated in [3, 15, 22, 25, 26]. For $0 \leq \alpha < 1$, $\mathcal{S}^*(\alpha)$ consists of only univalent functions while for other values of α, the class contains non-univalent functions. Other classes can be unified in a similar manner by subordination.

Motivated by the works of Lewandowski, Miller and Złotkiewicz [5], several authors [7, 8, 11, 15, 13, 14, 17, 18, 19, 23, 27] have investigated the functions f for which $zf'(z)/f(z)\cdot(\alpha zf''(z)/f'(z)+1)$ lies in certain region in the right half-plane. For $\alpha \geq 0$ and $\beta < 1$, Ravichandran *et al.* [21] have shown that a function f of the form $f(z) = z + a_{n+1}z^{n+1} + \cdots$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right) > \alpha\beta\left(\beta+\frac{n}{2}-1\right)+\beta-\frac{\alpha n}{2} \tag{1.3}$$

is starlike of order β. In the first result of Theorem 2.1, we obtain the corresponding result for $f \in \mathcal{A}_{n,b}$.

For function p of the form $p(z) = 1+p_1z+p_2z^2+\cdots$, Nunokawa *et al.* [12] showed that for certain analytic function w, with $w(0) = \alpha$, $\alpha p^2(z)+\beta zp'(z) \prec$

$w(z)$ implies $\operatorname{Re} p(z) > 0$, where $\beta > 0$, $\alpha \geq -\beta/2$. See also [20]. Lemma 2.6 investigates the conditions for similar class of functions.

For complex numbers β and γ, the differential subordination

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z),$$

where q is analytic and h is univalent with $q(0) = h(0)$, is popularly known as Briot-Bouquet differential subordination. This particular differential subordination has a significant number of important applications in the theory of analytic functions (for details see [9]). The importance of Briot-Bouquet differential subordination inspired many researchers to work in this area and many generalizations and extensions of the Briot-Bouquet differential subordination have recently been obtained. Ali *et al.* [1] obtained several results related to the Briot-Bouquet differential subordination. In Lemmas 2.2 and 2.5, the Briot-Bouquet differential subordination is investigated for functions with fixed second coefficient.

2. Subordinations for starlikeness and univalence

For $\beta \neq 1$, Theorem 2.1 provides several sufficient conditions for $f \in \mathcal{S}^*(\beta)$; in particular, for $0 \leq \beta < 1$, these are sufficient conditions for starlikeness of order β. Theorem 2.2 is the meromorphic analogue of Theorem 2.1. Theorem 2.3 gives sufficient conditions for the subordination $f'(z) \prec (1+(1-2\beta)z)/(1-z)$ to hold. For $\beta = 0$, this latter condition is sufficient for the close-to-convexity and hence univalence of the function f.

Theorem 2.1. *Let $\alpha \geq 0$, $\beta \geq 0$, $\beta \neq 1$, and $0 < \mu = nb \leq 2$. Let δ_1, δ_2, δ_3 and δ_4 be given by*

$$\delta_1 = -\frac{\alpha}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + (1-\alpha)\beta + \alpha\beta^2,$$
$$\delta_2 = -\frac{1}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + \beta,$$
$$\delta_3 = \begin{cases} \frac{-\alpha\beta}{2(1-\beta)}\left(n + \frac{2-\mu}{2+\mu}\right) + \beta, & if \quad 0 \leq \beta \leq \frac{1}{2}, \\ \frac{-\alpha}{2\beta}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + \beta, & if \quad \frac{1}{2} \leq \beta, \end{cases}$$
$$\delta_4 = \begin{cases} \frac{-\beta}{2(1-\beta)}\left(n + \frac{2-\mu}{2+\mu}\right), & if \quad 0 \leq \beta < \frac{1}{2}, \\ \frac{-1}{2\beta}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right), & if \quad \frac{1}{2} \leq \beta. \end{cases}$$

If $f \in \mathcal{A}_{n,b}$ satisfies one of the following subordinations

$$\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right) \prec \frac{1+(1-2\delta_1)z}{1-z}, \tag{2.1}$$
$$\frac{zf'(z)}{f(z)}\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{1+(1-2\delta_2)z}{1-z}, \tag{2.2}$$
$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+(1-2\delta_3)z}{1-z}, \tag{2.3}$$
$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec -\frac{2\delta_4 z}{1-z} \tag{2.4}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

Our next theorem gives sufficient conditions for meromorphic functions to be starlike in the punctured unit disk $\mathbb{D}^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$. Precisely, we consider the class $\Sigma_{n,b}$ of all analytic functions defined on $\mathbb{D}^*$ of the form

$$f(z) = \frac{1}{z} + bz^n + a_{n+1}z^{n+1} + \cdots \quad (b \leq 0).$$

Theorem 2.2. *Let $\alpha \geq 0$, $\beta \geq 0$, $\beta \neq 1$, and $0 < \mu = -(n+1)b \leq 2$. Let δ_1, δ_2, δ_3 and δ_4 be given by*

$$\delta_1 = -\frac{\alpha}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + (1-\alpha)\beta + \alpha\beta^2,$$
$$\delta_2 = -\frac{1}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + \beta,$$
$$\delta_3 = \begin{cases} \frac{-\alpha\beta}{2(1-\beta)}\left(n + \frac{2-\mu}{2+\mu}\right) + \beta, & if \quad 0 \leq \beta \leq \frac{1}{2}, \\ \frac{-\alpha}{2\beta}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + \beta, & if \quad \frac{1}{2} \leq \beta, \end{cases}$$
$$\delta_4 = \begin{cases} \frac{-\beta}{2(1-\beta)}\left(n + \frac{2-\mu}{2+\mu}\right), & if \quad 0 \leq \beta < \frac{1}{2}, \\ \frac{-1}{2\beta}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right), & if \quad \frac{1}{2} \leq \beta. \end{cases}$$

If $f \in \Sigma_{n,b}$ satisfies one of the following subordinations

$$\frac{zf'(z)}{f(z)}\left(2\alpha\frac{zf'(z)}{f(z)} - \alpha\frac{zf''(z)}{f'(z)} - 1\right) \prec \frac{1+(1-2\delta_1)z}{1-z}, \tag{2.5}$$
$$-\frac{zf'(z)}{f(z)}\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{1+(1-2\delta_2)z}{1-z}, \tag{2.6}$$
$$\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - (1+\alpha)\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\delta_3)z}{1-z}, \tag{2.7}$$
$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec -\frac{2\delta_4 z}{1-z} \tag{2.8}$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

Theorem 2.3. *Let $\alpha \geq 0$, $\beta \geq 0$, $\beta \neq 1$, and $0 < \mu = (n+1)b \leq 2$. Let δ_1, δ_2, δ_3 and δ_4 be given as in Theorem 2.1. If $f \in \mathcal{A}_{n,b}$ satisfies one of the following subordinations*

$$f'(z)\left[\alpha\left(\frac{zf''(z)}{f'(z)} + f'(z) - 1\right) + 1\right] \prec \frac{1+(1-2\delta_1)z}{1-z}, \tag{2.9}$$

$$f'(z) + zf''(z) \prec \frac{1+(1-2\delta_2)z}{1-z}, \tag{2.10}$$

$$\alpha\frac{zf''(z)}{f'(z)} + f'(z) \prec \frac{1+(1-2\delta_3)z}{1-z}, \tag{2.11}$$

$$\frac{zf''(z)}{f'(z)} \prec -\frac{2\delta_4 z}{1-z} \tag{2.12}$$

then

$$f'(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

The proof of these theorems follows from the following series of lemmas.

Lemma 2.1. *For $\alpha \geq 0$, $\beta \geq 0$, $\beta \neq 1$, $\gamma > 0$, and $0 < \mu \leq 2$, let*

$$\delta := -\frac{\gamma}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + (1-\alpha)\beta + \alpha\beta^2.$$

If $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$(1-\alpha)p(z) + \alpha p^2(z) + \gamma z p'(z) \prec \frac{1+(1-2\delta)z}{1-z}, \tag{2.13}$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Let $0 \leq \beta < 1$. Note that δ given in the hypothesis clearly satisfies $\delta < 1$. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = (p(z) - \beta)/(1-\beta)$. Then q is analytic and $(1-\beta)q(z) + \beta = p(z)$. By using this, the inequality (2.13) can be written as

$$\operatorname{Re}\Big[(1-\beta)(1-\alpha+2\alpha\beta)q(z) + \alpha(1-\beta)^2 q^2(z) \\ +\gamma(1-\beta)zq'(z) + (1-\alpha)\beta + \alpha\beta^2 - \delta\Big] > 0.$$

Define the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\psi(r, s) = (1-\beta)(1-\alpha+2\alpha\beta)r + \alpha(1-\beta)^2 r^2 + \gamma(1-\beta)s + (1-\alpha)\beta + \alpha\beta^2 - \delta.$$

For $\rho \in \mathbb{R}$, $n \geq 1$ and σ satisfying (1.2), it follows that

$$\begin{aligned}
&\operatorname{Re}\psi(i\rho, \sigma) \\
&= \operatorname{Re}\left[(1-\beta)(1-\alpha+2\alpha\beta)i\rho - \alpha(1-\beta)^2\rho^2 + \gamma(1-\beta)\sigma + (1-\alpha)\beta + \alpha\beta^2 - \delta\right] \\
&= \gamma(1-\beta)\sigma - \alpha(1-\beta)^2\rho^2 + (1-\alpha)\beta + \alpha\beta^2 - \delta \\
&\leq \gamma(1-\beta)\left[-\frac{1}{2}\left(n + \frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right] - \alpha(1-\beta)^2\rho^2 + (1-\alpha)\beta + \alpha\beta^2 - \delta \\
&= -\frac{\gamma}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right)(1+\rho^2) - \alpha(1-\beta)^2(\rho^2+1) + \alpha(1-\beta)^2 \\
&\quad + (1-\alpha)\beta + \alpha\beta^2 - \delta \\
&= -(1+\rho^2)\left[\frac{\gamma}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + \alpha(1-\beta)^2\right] \\
&\quad + \alpha(1-\beta)^2 + (1-\alpha)\beta + \alpha\beta^2 - \delta \\
&\leq -\frac{\gamma}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + (1-\alpha)\beta + \alpha\beta^2 - \delta.
\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho, \sigma) \leq 0$, or $\psi \in \Psi_{\mu,n}$. By Theorem 1.1, $\operatorname{Re} q(z) > 0$ or equivalently $\operatorname{Re} p(z) > \beta$. For $\beta > 1$, the proof is similar. □

Lemma 2.2. *For $\beta \geq 0$, $\beta \neq 1$, $\gamma > 0$, and $0 < \mu \leq 2$, let*

$$\delta := -\frac{\gamma}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + \beta.$$

If $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$p(z) + \gamma z p'(z) \prec \frac{1+(1-2\delta)z}{1-z},$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Replace $\alpha = 0$ in Lemma 2.1 to yield the result. □

Lemma 2.3. *For $\alpha > 0$, $\beta \geq 0$, $\beta \neq 1$, and $0 < \mu \leq 2$, let*

$$\delta = \begin{cases} \frac{-\alpha\beta}{2(1-\beta)} \left(n + \frac{2-\mu}{2+\mu}\right) + \beta := \delta_2, & if \quad 0 \leq \beta \leq \frac{1}{2}, \\ \frac{-\alpha}{2\beta}(1-\beta) \left(n + \frac{2-\mu}{2+\mu}\right) + \beta := \delta_1, & if \quad \frac{1}{2} \leq \beta. \end{cases}$$

If the function $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$p(z) + \alpha \frac{zp'(z)}{p(z)} \prec \frac{1+(1-2\delta)z}{1-z} \tag{2.14}$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Let $0 \leq \beta < 1$. As in the proof of Lemma 2.1, let $q : \mathbb{D} \to \mathbb{C}$ be given by $q(z) = (p(z) - \beta)/(1-\beta)$. Then inequality (2.14) can be written as

$$\operatorname{Re}\left[(1-\beta)q(z) + \beta + \frac{\alpha(1-\beta)}{(1-\beta)q(z)+\beta} zq'(z) - \delta\right] > 0. \tag{2.15}$$

Define the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\psi(r,s) = (1-\beta)r + \frac{\alpha(1-\beta)}{(1-\beta)r+\beta} s + \beta - \delta.$$

Then $\operatorname{Re}\psi(q(z), zq'(z)) > 0$ and $\operatorname{Re}\psi(1,0) > 0$. We now show that $\psi \in \Psi_{\mu,n}$. For $\rho \in \mathbb{R}$, $n \geq 1$ and σ satisfying (1.2), it follows that

$$\begin{aligned} \operatorname{Re}\psi(i\rho,\sigma) &= \operatorname{Re}\left[(1-\beta)i\rho + \frac{\alpha(1-\beta)}{(1-\beta)i\rho+\beta}\sigma + \beta - \delta\right] \\ &= \operatorname{Re}\left[(1-\beta)i\rho + \frac{\alpha\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\sigma - \frac{\alpha(1-\beta)^2 i\rho}{\beta^2+(1-\beta)^2\rho^2}\sigma + \beta - \delta\right] \\ &= \frac{\alpha\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\sigma + \beta - \delta \\ &\leq \frac{\alpha\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\left[-\frac{1}{2}\left(n + \frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right] + \beta - \delta \\ &= -\frac{\alpha\beta}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right)\left(\frac{1+\rho^2}{\beta^2+(1-\beta)^2\rho^2}\right) + \beta - \delta. \end{aligned}$$

For $1/2 \leq \beta$, the expression

$$\frac{1+\rho^2}{\beta^2+(1-\beta)^2\rho^2}$$

attains minimum at $\rho = 0$ and therefore

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma) &\leq -\frac{\alpha\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{\beta^2}+\beta-\delta_1\\ &= \frac{-\alpha}{2\beta}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\beta-\delta_1.\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$, or $\psi \in \Psi_{\mu,n}$.

For $0 \leq \beta \leq 1/2$,

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma) &\leq -\frac{\alpha\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{(1-\beta)^2}+\beta-\delta_2\\ &= \frac{-\alpha\beta}{2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right)+\beta-\delta_2.\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$ or $\psi \in \Psi_{\mu,n}$. Thus Theorem 1.1 implies $\operatorname{Re} q(z) > 0$ or equivalently $\operatorname{Re} p(z) > \beta$. The proof of the case $\beta > 1$ is similar. $\square$

Lemma 2.4. *For $\beta \geq 0$, $\beta \neq 1$ and $0 < \mu \leq 2$, let*

$$\delta = \begin{cases} \frac{-\beta}{2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right), & if \quad 0 \leq \beta < \frac{1}{2},\\ \frac{-1}{2\beta}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right), & if \quad \frac{1}{2} \leq \beta.\end{cases}$$

If the function $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$\frac{zp'(z)}{p(z)} \prec -\frac{2\delta z}{1-z},$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. We consider the case $0 \leq \beta < 1$. The case $\beta > 1$ is similar. Let $q(z) = (p(z)-\beta)/(1-\beta)$ or $(1-\beta)q(z)+\beta = p(z)$. Then

$$\frac{zp'(z)}{p(z)} = \frac{(1-\beta)zq'(z)}{(1-\beta)q(z)+\beta}. \tag{2.16}$$

Define $\psi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\psi(r,s) = \frac{(1-\beta)s}{(1-\beta)r+\beta} - \delta.$$

Then $\psi(r,s)$ is continuous on $(\mathbb{C} - \{-\beta/(1-\beta)\}) \times \mathbb{C}$. For $\rho \in \mathbb{R}$, $n \geq 1$ and σ satisfying (1.2), it follows that

$$\begin{aligned}
\operatorname{Re}\psi(i\rho,\sigma) &= \operatorname{Re}\left(\frac{(1-\beta)}{(1-\beta)i\rho+\beta}\sigma - \delta\right) \\
&= \operatorname{Re}\left(\frac{\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\sigma - \frac{(1-\beta)^2 i\rho}{\beta^2+(1-\beta)^2\rho^2}\sigma - \delta\right) \\
&= \frac{\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\sigma - \delta \\
&\leq \frac{\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\left[-\frac{1}{2}\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right] - \delta \\
&= -\frac{\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\left(\frac{1+\rho^2}{\beta^2+(1-\beta)^2\rho^2}\right) - \delta.
\end{aligned}$$

For $1/2 \leq \beta$, the expression

$$\frac{1+\rho^2}{\beta^2+(1-\beta)^2\rho^2}$$

attains its minimum at $\rho = 0$ and therefore

$$\begin{aligned}
\operatorname{Re}\psi(i\rho,\sigma) &\leq -\frac{\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{\beta^2} - \delta \\
&= \frac{-1}{2\beta}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right) - \delta.
\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$, or $\psi \in \Psi_{\mu,n}$.

For $0 \leq \beta \leq 1/2$,

$$\begin{aligned}
\operatorname{Re}\psi(i\rho,\sigma) &\leq -\frac{\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{(1-\beta)^2} - \delta \\
&= \frac{-\beta}{2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right) - \delta.
\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$, or $\psi \in \Psi_{\mu,n}$. Thus Theorem 1.1 implies $\operatorname{Re} q(z) > 0$ or equivalently $\operatorname{Re} p(z) > \beta$. $\square$

Lemma 2.5. *For $\alpha > 0$, $\beta \geq 0$, $\beta \neq 1$, $\gamma > -\alpha\beta$ and $0 < \mu \leq 2$, let*

$$\delta = \begin{cases} \frac{-1}{2}\frac{(1-\beta)}{(\alpha\beta+\gamma)}\left(n+\frac{2-\mu}{2+\mu}\right)+\beta, & if \quad \gamma \geq \alpha(1-2\beta), \\ \frac{-1}{2}\frac{(\alpha\beta+\gamma)}{\alpha^2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right)+\beta, & if \quad \gamma \leq \alpha(1-2\beta). \end{cases}$$

If the function $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$p(z) + \frac{zp'(z)}{\alpha p(z)+\gamma} \prec \frac{1+(1-2\delta)z}{1-z},$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. We consider the case $0 \leq \beta < 1$. The case $\beta > 1$ is similar. Define $q(z) = (p-\beta)/(1-\beta)$ or $(1-\beta)q+\beta = p(z)$. Then

$$p(z) + \frac{zp'(z)}{\alpha p(z)+\gamma} = (1-\beta)q(z)+\beta+\frac{(1-\beta)}{\alpha[(1-\beta)q(z)+\beta]+\gamma}zq'(z). \quad (2.17)$$

Define $\psi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\psi(r,s) = (1-\beta)r + \frac{(1-\beta)}{\alpha(1-\beta)r+\alpha\beta+\gamma}s+\beta-\delta.$$

Thus $\psi(r,s)$ is continuous and for $\rho \in \mathbb{R}$, $n \geq 1$ and σ satisfying (1.2), it follows that

$$\begin{aligned} \operatorname{Re}\psi(i\rho,\sigma) &= \operatorname{Re}\left[(1-\beta)i\rho + \frac{(1-\beta)}{\alpha(1-\beta)i\rho+\alpha\beta+\gamma}\sigma+\beta-\delta\right] \\ &= \frac{(1-\beta)(\alpha\beta+\gamma)}{(\alpha\beta+\gamma)^2+\alpha^2(1-\beta)^2\rho^2}\sigma+\beta-\delta \\ &\leq \frac{(1-\beta)(\alpha\beta+\gamma)}{(\alpha\beta+\gamma)^2+\alpha^2(1-\beta)^2\rho^2}\left[-\frac{1}{2}\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right]+\beta-\delta \\ &= \frac{-1}{2}(1-\beta)(\alpha\beta+\gamma)\left(n+\frac{2-\mu}{2+\mu}\right)\left(\frac{1+\rho^2}{(\alpha\beta+\gamma)^2+\alpha^2(1-\beta)^2\rho^2}\right)+\beta-\delta \end{aligned}$$

For $\gamma \leq \alpha(1-2\beta)$,

$$\begin{aligned} \operatorname{Re}\psi(i\rho,\sigma) &\leq \frac{-1}{2}(1-\beta)(\alpha\beta+\gamma)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{\alpha^2(1-\beta)^2}+\beta-\delta \\ &= \frac{-1}{2}\frac{(\alpha\beta+\gamma)}{\alpha^2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right)+\beta-\delta. \end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$, or $\psi \in \Psi_{\mu,n}$.

For $\gamma \geq \alpha(1-2\beta)$, the expression

$$\frac{1+\rho^2}{(\alpha\beta+\gamma)^2+\alpha^2(1-\beta)^2\rho^2}$$

attains minimum at $\rho = 0$ and therefore

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma) &\leq \frac{-1}{2}(1-\beta)(\alpha\beta+\gamma)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{(\alpha\beta+\gamma)^2}+\beta-\delta\\ &= \frac{-1}{2}\frac{(1-\beta)}{(\alpha\beta+\gamma)}\left(n+\frac{2-\mu}{2+\mu}\right)+\beta-\delta.\end{aligned}$$

Thus $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$, or $\psi \in \Psi_{\mu,n}$, and result follows from Theorem 1.1. □

Lemma 2.6. *For $\beta \geq 0$, $\beta \neq 1$, $\gamma > 0$, and $0 < \mu \leq 2$. If the function $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination*

$$p^2(z)+\gamma z p'(z) \prec \frac{1+(1-2\delta)z}{1-z}, \tag{2.18}$$

where

$$\delta := -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\beta^2,$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. We consider the case $0 \leq \beta < 1$. The case $\beta > 1$ is similar. Define $q(z) = (p(z)-\beta)/(1-\beta)$ or $(1-\beta)q(z)+\beta = p(z)$. Using this it can be shown that inequality (2.18) can be written as

$$\operatorname{Re}\left[((1-\beta)q(z)+\beta)^2+\gamma(1-\beta)zq'(z)-\delta\right] > 0.$$

Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ be defined by

$$\psi(r,s) = [(1-\beta)r+\beta]^2+\gamma(1-\beta)s-\delta.$$

For $\rho \in \mathbb{R}$, $n \geq 1$ and σ satisfying (1.2), it follows that

$$\begin{aligned}
&\operatorname{Re}\psi(i\rho,\sigma)\\
&= \operatorname{Re}\left[((1-\beta)i\rho+\beta)^2+\gamma(1-\beta)\sigma-\delta\right]\\
&= -(1-\beta)^2\rho^2+\beta^2+\gamma(1-\beta)\sigma-\delta\\
&\leq \gamma(1-\beta)\left[-\frac{1}{2}\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right]+\beta^2-(1-\beta)^2\rho^2-\delta\\
&= -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)-(1-\beta)^2(\rho^2+1)+(1-\beta)^2+\beta^2-\delta\\
&= -(1+\rho^2)\left[\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+(1-\beta)^2\right]+(1-\beta)^2+\beta^2-\delta\\
&\leq -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\beta^2-\delta.
\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$, or $\psi \in \Psi_{\mu,n}$, and Theorem 1.1 implies $\operatorname{Re} q(z) > 0$ or equivalently $\operatorname{Re} p(z) > \beta$. □

Proof of Theorem 2.1. For a given function $f \in \mathcal{A}_{n,b}$, let the function $p : \mathbb{D} \to \mathbb{C}$ be defined by $p(z) = zf'(z)/f(z)$. Then computation shows that $p(z) = 1 + nbz^n + \cdots \in \mathcal{H}_{\mu,n}$ where $\mu = nb$. Further calculations yield

$$\begin{aligned}
&\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right) = (1-\alpha)p(z)+\alpha p^2(z)+\alpha zp'(z),\\
&\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) = p(z)+zp'(z),\\
&(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) = p(z)+\alpha\frac{zp'(z)}{p(z)},\\
&1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)}.
\end{aligned}$$

Hence the result follows from Lemmas 2.1–2.4. □

Proof of Theorem 2.2. Let $f \in \Sigma_{n,b}$, and define the function $p : \mathbb{D} \to \mathbb{C}$ be defined by $p(0) = 1$ and $p(z) = -zf'(z)/f(z)$ for $z \in \mathbb{D}^*$. Then $p(z) = 1-(n+1)bz^{n+1}+\cdots \in \mathcal{H}_{\mu,n}$ with $\mu = -(n+1)b$. Simple computations shows

that

$$\frac{zf'(z)}{f(z)}\left(2\alpha\frac{zf'(z)}{f(z)}-\alpha\frac{zf''(z)}{f'(z)}-1\right)=(1-\alpha)p(z)+\alpha p^2(z)+\alpha zp'(z),$$
$$-\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)=p(z)+zp'(z),$$
$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-(1+\alpha)\frac{zf'(z)}{f(z)}=p(z)+\alpha\frac{zp'(z)}{p(z)},$$
$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}=\frac{zp'(z)}{p(z)}.$$

Hence the result follows from Lemmas 2.1–2.4. □

Proof of Theorem 2.3. For $f\in\mathcal{A}_{n,b}$, let the function $p:\mathbb{D}\to\mathbb{C}$ be defined by $p(z)=f'(z)$. Then $p(z)=1+(n+1)bz^n+(n+2)a_{n+2}z^{n+1}+\cdots\in\mathcal{H}_{\mu,n}$ with $\mu=(n+1)b$. Also, we have the following:

$$f'(z)\left(\alpha\left(\frac{zf''(z)}{f'(z)}+f'(z)-1\right)+1\right)=(1-\alpha)p(z)+\alpha p^2(z)+\alpha zp'(z),$$
$$f'(z)+\alpha zf''(z)=p(z)+\alpha zp'(z),$$
$$\alpha\frac{zf''(z)}{f'(z)}+f'(z)=p(z)+\alpha\frac{zp'(z)}{p(z)},$$
$$\frac{zf''(z)}{f'(z)}=\frac{zp'(z)}{p(z)}.$$

Hence the result follows from Lemmas 2.1–2.4. □

Remark 2.1.

(i) For $\beta=0$, the condition (2.9)–(2.12) gives a sufficient condition for close-to-convexity and hence for univalence.

(ii) If $\mu=2$, result (2.1) reduces to [21, Theorem 2.1]. If $\mu=2$, and $f'(z)$ is considered as $f(z)/z$, result (2.10) reduces to [21, Theorem 2.4]. Inequality (2.11) reduces to [24, Theorem 2, p. 182] in the case when $\mu=2$, $n=1$ and $\beta=1/2$. Furthermore, if $\mu=2$, $n=1$ and $\beta=(\alpha+1)/2$, result (2.12) reduces to [16, Theorem 1].

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