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Majorization for Certain Classes of Analytic Functions of Complex Order Associated with the Dziok-Srivastava and the Srivastava-Wright Convolution Operators *

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Abstract

The main object of this present paper is to investigate the problem of majorization for certain classes of analytic functions of complex order associated associated with the Dziok-Srivastava and the Srivastava-Wright convolution operators. Moreover we point out some new or known consequences of our main result.

Keywords and Phrases: *Analytic functions, Starlike and convex functions of complex order, Qusai-subordination, Majorization problems, Hadamard product (convolution), Dziok-Srivastava operator , Srivastava-Wright convolution operator.*

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1. Introduction

Let $\mathcal{S}$ be the class of functions which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

For given $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n \in \mathcal{S}$ the Hadamard product of f and g is denoted by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \tag{1.2}$$

note that $f * g \in \mathcal{S}$ which are analytic in the open disc $\mathbb{U}$.

For two analytic functions $f, g \in \mathcal{S}$ we say that f is subordinate to g denoted by $f \prec g$ if there exists a Schwar'z function $\omega(z)$ which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$ and $z \in \mathbb{U}$.

Note that, if the function g is univalent in $\mathbb{U}$, due to Miller and Mocanu [13] we have

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

If f and g are analytic functions in $\mathbb{U}$, following MacGregor [12], we say that f is majorized by g in $\mathbb{U}$ that is $f(z) \ll g(z), (z \in \mathbb{U})$ if there exists a function $\phi(z)$, analytic in $\mathbb{U}$, such that

$$|\phi(z)| < 1 \; and \; f(z) = \phi(z) g(z), \; z \in \mathbb{U}.$$

It is interested to note that the notation of majorization is closely related to the concept of quasi-subordination between analytic functions.

Recently Dziok and Srivastava [4, 5] defined the linear operator of a function $f(z)$, denoted by $H_m^l[\alpha_1] f(z)$, is defined by

$$H_m^l(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) : \mathcal{S} \to \mathcal{S}$$

such that

$$\begin{aligned} H_m^l[\alpha_1]f(z) &\equiv H(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z) \\ &= z\ {}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z)*f(z) \\ H_m^l[\alpha_1]f(z) &= z+\sum_{n=2}^{\infty}\Gamma(n)\ a_n\ z^n, \end{aligned} \tag{1.3}$$

where

$$\Gamma(n)=\frac{(\alpha_1)_{n-1}\ldots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_m)_{n-1}}\ \frac{1}{(n-1)!}. \tag{1.4}$$

It is easy to verify from (1.3) that

$$z(H_m^l[\alpha_1]f(z))'=\alpha_1H_m^l[\alpha_1+1]f(z)-(\alpha_1-1)H_m^l[\alpha_1]f(z). \tag{1.5}$$

Note that if $l=2$ and $m=1$ with $\alpha_1=1;\alpha_2=1;\beta_1=1$ then $H[\alpha_1]f(z)=f(z)$.

It is of interest to note that the following are the special cases of the Dziok-Srivastava linear operator.

Remark 1. *For $f\in\mathcal{S}$, $H_1^2(a,1;c)f(z)=\mathcal{L}(a,c)f(z)=z+\sum\limits_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}a_nz^n$ was considered by Carlson and Shaffer [3].*

Remark 2. *By using the Gaussian hypergeometric function given by*

$${}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z),$$

Hohlov [8] introduced a generalized convolution operator $H_{a,b,c}$ as

$$H_{a,b,c}f(z)=z_2F_1(a,b,c;z)*f(z),$$

contains as special cases most of the known linear integral or differential operators.

Remark 3. *For $f\in\mathcal{S}, H_1^2(\delta+1,1;1)f(z)=\frac{z}{(1-z)^{\delta+1}}*f(z)=\mathcal{D}^\delta f(z),(\delta>-1)$ the $\mathcal{D}^\delta f'(z)=z+\sum_{n=2}^{\infty}\begin{pmatrix}\delta+n-1\\ n-1\end{pmatrix}a_nz^n$, was introduced by Ruscheweyh [18].*

Remark 4. *For* $f \in \mathcal{S}, H_1^2(c+1,1;c+2)f(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt = \mathcal{J}_c f(z)$ *where* $c > -1$. *The operator* $\mathcal{J}_c$ *was introduced by Bernardi [2]. In particular, the operator* $\mathcal{J}_1$ *was studied earlier by Libera [10] and Livingston [11].*

Remark 5. *For* $f \in \mathcal{S}, H_1^2(2,1;2-\lambda)f(z) = \Gamma(2-\lambda)z^\lambda \mathcal{D}_z^\lambda f(z) = \Omega^\lambda f(z), \quad \lambda \notin \mathbb{N}\setminus\{1\}$. *The operator* Ω^λ *was introduced by Srivastava-Owa [19] and* Ω^λ *is also called Srivastava-Owa fractional derivative operator, where* $\mathcal{D}_z^\lambda f(z)$ *denotes the fractional derivative of* $f(z)$ *of order* λ, *studied by Owa [17].*

Geometric Function Theory also contains systematic investigations of various analytic function classes associated with a *further* generalization of the Dziok-Srivastava convolution operator, which is popularly known as the Wright-Srivastava convolution operator defined by using the Fox-Wright generalized hypergeometric function (see, for details, [9] and [20]; see also [23] and the references cited in each of these recent works including [9] and [20]). Following Dziok and Srivastava [4], using Wright's generalized hypergeometric function [21], Dziok and Raina [6] defined another linear operator given by

$$\mathcal{W}[\alpha_1]f(z) = z + \sum_{n=2}^{\infty} \sigma_n \, a_n z^n, \quad z \in \mathbb{U}, \tag{1.6}$$

where

$$\sigma_n(\alpha_1) = \frac{\Theta \, \Gamma(\alpha_1 + A_1(n-1))\ldots\Gamma(\alpha_l + A_l(n-1))}{(n-1)!\Gamma(\beta_1 + B_1(n-1))\ldots\Gamma(\beta_m + B_m(n-1))}\,, \tag{1.7}$$

and Θ is given by $\Theta = \left(\prod_{t=0}^{l}\Gamma(\alpha_t)\right)^{-1}\left(\prod_{t=0}^{m}\Gamma(\beta_t)\right)$. Here, presumably, $\Gamma(a)$ denotes a value of the gamma function. It is easy to verify from (1.6) that

$$zA_1(\mathcal{W}[\alpha_1]f(z))' = \alpha_1\mathcal{W}[\alpha_1+1]f(z) - (\alpha_1 - A_1)\mathcal{W}[\alpha_1]f(z). \tag{1.8}$$

For $A_l = B_m = 1$, the Dziok-Raina operator $\mathcal{W}[\alpha_1]f(z)$ yields the Dziok-Srivastava operator [6], and for the suitable choices of l, m in turn it includes various operators defined by Hohlov [8], Ruscheweyh [18], Carlson and Shaffer [3] and the integral operators introduced by Bernardi [2] and Libera [10] as mentioned in Remarks 1 to 5.

Using the Wright hypergeometric linear operator given by (1.6) , we now introduce the following new subclass of $\mathcal{S}$.

Definition 1. *A function $f(z) \in \mathcal{S}$ is said to in the class $\mathcal{S}_m^l([\alpha_1]; A, B; \gamma)$, if and only if*

$$1 + \frac{1}{\gamma}\left[\frac{z(\mathcal{W}[\alpha_1]f(z))'}{\mathcal{W}[\alpha_1]f(z)} - 1\right] \prec \frac{1+Az}{1+Bz}, \tag{1.9}$$

where $z \in \mathbb{U}$, $-1 \le B < A \le 1$, and $\gamma \in \mathbb{C}\backslash\{0\}$.

For simplicity, we put

$$\mathcal{S}_m^l([\alpha_1]; A, B; \gamma) = \mathcal{S}_m^l([\alpha_1]; 1, -1; \gamma),$$

where $\mathcal{S}_m^l([\alpha_1]; 1, -1; \gamma)$ denote the class of functions $f \in \mathcal{S}$ satisfying the following inequality:

$$\Re\left(1 + \frac{1}{\gamma}\left[\frac{z(\mathcal{W}[\alpha_1]f(z))'}{\mathcal{W}[\alpha_1]f(z)} - 1\right]\right) > 0. \tag{1.10}$$

Clearly, we have the following relationships:

1. For $A_i = B_j = 1$ $(i = \overline{1,l}; j = \overline{1,m})$, $\mathcal{S}_m^l([\alpha_1]; 1, -1; \gamma) :\equiv \mathcal{H}_m^l([\alpha_1]; \gamma)$ $(\gamma \in \mathbb{C} \setminus \{0\})$ [14].

2. For $l = 2$, $m = 1$, and $A_i = B_j = 1$ $(i = \overline{1,l}; j = \overline{1,m})$, $\mathcal{S}_1^2(\alpha_1 = \beta_1; \alpha_2 = 1; 1, -1; \gamma) :\equiv S(\gamma)$ $(\gamma \in \mathbb{C} \setminus \{0\})$ [16].

3. For $l = 2$, $m = 1$, and $A_i = B_j = 1$ $(i = \overline{1,l}; j = \overline{1,m})$, $\mathcal{S}_1^2(\alpha_1 = 2; \beta_1 = 1; \alpha_2 = 1; 1, -1; \gamma) :\equiv K(\gamma)$ $(\gamma \in \mathbb{C} \setminus \{0\})$ [22].

4. For $l = 2$, $m = 1$, and $A_i = B_j = 1$ $(i = \overline{1,l}; j = \overline{1,m})$, $\mathcal{S}_1^2(\alpha_1 = \beta_1; \alpha_2 = 1; 1, -1; 1-\alpha) :\equiv S^*(\alpha)$, $(0 \le \alpha < 1)$.

Moreover $S^*(\alpha)$, denotes the class of starlike functions of order α in $\mathbb{U}$. Majorization problems for the class $S^* = S^*(0)$ had been investigated by MacGregor [12], recently Altintas et al. [1] investigated a majorization problem for the class $S(\gamma)$. Very recently Goyal and Goswami [7] generalized these results for the fractional operator. In this paper we investigated a majorization problem for the class $\mathcal{S}_m^l([\alpha_1]; A, B; \gamma)$, and give some special cases of our result.

2 A MAJORIZATION PROBLEM FOR THE CLASS $\mathcal{S}_m^l([\alpha_1]; A, -B; \gamma)$

Theorem 1. *Let the function $f(z) \in \mathcal{S}$, and suppose that $g(z) \in \mathcal{S}_m^l([\alpha_1]; A, B; \gamma)$. If $\mathcal{W}[\alpha_1]f(z)$ is majorized by $\mathcal{W}[\alpha_1]g(z)$ in $\mathbb{U}$ then*

$$|\mathcal{W}[\alpha_1+1]f(z)| \leq |\mathcal{W}[\alpha_1+1]g(z)|, \quad |z| \leq r_1, \tag{2.1}$$

where r_1 is smallest the positive root of the equation

$$|A_1\gamma(A-B)+\alpha_1 B|r^3-[|\alpha_1|+2|A_1||B|]r^2-[|A_1\gamma(A-B)+\alpha_1 B|+2|A_1|]r|\alpha_1| = 0, \tag{2.2}$$

where $-1 \leq B < A \leq 1$, $|\alpha_1| \geq |A_1\gamma(A-B)+\alpha_1 B|$ and $\gamma \in \mathbb{C} \setminus \{0\}$.

Proof. Since $g \in \mathcal{S}_m^l([\alpha_1]; A, B; \gamma)$,we find from (1.10 that

$$1+\frac{1}{\gamma}\left(\frac{z(\mathcal{W}[\alpha_1]g(z))'}{\mathcal{W}[\alpha_1]g(z)}-1\right) = \frac{1+Aw(z)}{1+Bw(z)}, \tag{2.3}$$

where w is analytic in $\mathbb{U}$, with $w(0)$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$. From (2.3), we get

$$\frac{z(\mathcal{W}[\alpha_1]g(z))'}{\mathcal{W}[\alpha_1]g(z)} = \frac{1+[\gamma(A-B)+B]w(z)}{1+Bw(z)}. \tag{2.4}$$

Now, by applying the relation (1.8), in (2.4) we get

$$\frac{\mathcal{W}[\alpha_1+1]g(z)}{\mathcal{W}[\alpha_1]g(z)} = \frac{\alpha_1+[A_1\gamma(A-B)+\alpha_1 B]w(z)}{\alpha_1[1+Bw(z)]}, \tag{2.5}$$

which yields that,

$$|\mathcal{W}[\alpha_1]g(z)| = \frac{|\alpha_1|[1+|B|z|]}{|\alpha_1|-|A_1\gamma(A-B)+\alpha_1 B]||z|}\,|\mathcal{W}[\alpha_1+1]g(z)|\,. \tag{2.6}$$

Since $\mathcal{W}[\alpha_1]f(z)$ is majorized by $\mathcal{W}[\alpha_1]g(z)$ in $\mathbb{U}$ then $\mathcal{W}[\alpha_1]f(z) = \phi(z)\mathcal{W}[\alpha_1]g(z)$ and differentiating with respect to z we get

$$z(\mathcal{W}[\alpha_1]f(z))' = z\phi'(z)\mathcal{W}[\alpha_1]g(z) + z\phi(z)(\mathcal{W}[\alpha_1]g(z))'. \tag{2.7}$$

Noting that the Schwarz function $\phi(z)$ satisfies (cf. [15])

$$|\phi^{'}(z)| \leq \frac{1-|\phi(z)|^2}{1-|z|^2} \tag{2.8}$$

and using (1.8), (2.6) and (2.8) in (2.7), we have

$$|\mathcal{W}[\alpha_1+1]f(z)| \leq \left(|\phi(z)| + \left(\frac{1-|\phi(z)|^2}{1-|z|^2}\right)\frac{|A_1|[1+|B||z|]|z|}{|\alpha_1|-|A_1\gamma(A-B)+\alpha_1 B||z|}\right) |\mathcal{W}[\alpha_1+1]g(z)|. \tag{2.9}$$

Setting $|z|=r$ and $|\phi(z)|=\rho$, $0\leq\rho\leq 1$ leads us to the inequality

$$|\mathcal{W}[\alpha_1+1]f(z)| \leq \frac{\Phi(\rho)}{(1-r^2)[|\alpha_1|-|A_1\gamma(A-B)+\alpha_1 B|r]}|\mathcal{W}[\alpha_1+1]g(z)|, \tag{2.10}$$

where the function $\Phi(\rho)$ defined by

$$\Phi(\rho) = -|A_1|r[1+|B|r]\rho^2+(1-r^2)[|\alpha_1|-|A_1\gamma(A-B)+\alpha_1 B|r]\rho+|A_1|r[1+|B|r]$$

takes its maximum value at $\rho=1$ with with $r=r_1(\gamma,A,B)$, the smallest positive root of the equation (2.2).

Furthermore, if $0\leq\sigma\leq r_1$, then the function $\varphi(\rho)$ defined by

$$\varphi(\rho) = -|A_1|\sigma[1+|B|\sigma]\rho^2+(1-\sigma^2)[|\alpha_1|-|A_1\gamma(A-B)+\alpha_1 B|\sigma]\rho+|A_1|\sigma[1+|B|\sigma]$$

is an increasing function on $(0\leq\rho\leq 1)$ so that

$$\varphi(\rho) = (1-\sigma^2)[|\alpha_1|-|A_1\gamma(A-B)+\alpha_1 B|\sigma]+|A_1|\sigma[1+|B|\sigma],$$

$0\leq\rho\leq 1,\ 0\leq\sigma\leq r_1$. Therefore, from this fact, (2.10) gives the inequality (2.1).

Putting $A=1$, $B=-1$, $\gamma=(1-\alpha)cos\lambda e^{-i\lambda}$, $|\lambda|<\frac{\pi}{2}$; $(0\leq\alpha\leq 1)$, with $l=2, m=1$, $A_t=B_t=1$ and $\alpha_1=\alpha_2=1$; $\beta_1=1$ in Theorem 1, we have the following corollary:

Corollary 1. *Let the function $f(z)\in A$ and $g(z)\in S(\gamma)$ ($\gamma=(1-\alpha)cos\lambda e^{-i\lambda}$, $|\lambda|<\frac{\pi}{2}$; $0\leq\alpha\leq 1$). If*

$$|f'(z)|\leq|g'(z)|,\ |z|\leq r_2, \tag{2.11}$$

where

$$r_2=\frac{\delta-\sqrt{\delta^2-4|2(1-\alpha)cos\lambda e^{-i\lambda}-1|}}{2|2(1-\alpha)cos\lambda e^{-i\lambda}-1|} \tag{2.12}$$

and

$$\delta=|2(1-\alpha)cos\lambda e^{-i\lambda}-1|+3.$$

Further taking $A = 1$, $B = -1$, $l = 2, m = 1$, $A_t = B_t = 1$ and $\alpha_1 = \alpha_2 = 1$; $\beta_1 = 1$ in Theorem 1, we have the following corollary

Corollary 2. *Let the function $f(z) \in \mathcal{S}$ be analytic and univalent in the open unit disk $\mathbb{U}$ and suppose that $g(z) \in S(\gamma)$. If $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, then*

$$|f'(z)| \leq |g'(z)|, \ |z| \leq r_3,$$

where

$$r_3 := \frac{3 + |2\gamma - 1| - \sqrt{9 + 2|2\gamma - 1| + |2\gamma - 1|^2}}{2|2\gamma - 1|}$$

.

For $\gamma = 1$, Corollary 2 reduces to the following result:

Corollary 3. *[12] Let the function $f(z) \in \mathcal{S}$ be analytic and univalent in the open unit disk $\mathbb{U}$ and suppose that $g(z) \in S^* = S^*(0)$. If $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, then*

$$|f'(z)| \leq |g'(z)|, \ |z| \leq r_4,$$

where $r_4 := 2 - \sqrt{3}$.

Concluding Remarks: Further specializing the parameters l, m one can define the various other interesting subclasses of $\mathcal{S}_m^l([\alpha_1]; A, B; \gamma)$, involving the differential operators as stated in Remarks 1 to 5, and the result as in Theorem 1 and the corresponding corollaries as mentioned above can be derived easily. The details involved may be left as an exercise for the interested reader.

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