A New Version of The Stirling Formula*

Zheng Liu
Institute of Applied Mathematics, Faculty of Science
Anshan University of Science and Technology
Anshan 114044, Liaoning, China

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Abstract
A new version of the Stirling formula is given as

\[ n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \exp\left(\int_0^\infty \frac{1}{x} - \{x\} \, dx\right), \]

and it is applied to provide a new and more natural proof of a recent version due to L. C. Hsu.

Keywords and Phrases: Stirling formula, Wallis’ product formula, Infinite integral.

1. Introduction

Stirling formula and its different versions have a fascinating history. The classical form containing Bernoulli numbers which has been studied deeply and thoroughly in [1] and [2] where an infinite numbers of recurrence relations for the Bernoulli numbers are obtained.

It is very interesting that in the last decade Hsu in [3] has given a new version without using Bernoulli numbers as the following identity

\[ n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \exp\left(\sum_{k=n}^{\infty} \sum_{j=2}^{\infty} \frac{j-1}{2j(j+1)} (-1)^j \right), \]

\[ \sum_{k=n}^{\infty} \sum_{j=2}^{\infty} \frac{j-1}{2j(j+1)} (-1)^j, \tag{1} \]

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whose proof is elementary and simple in nature, and it is applied in [4] to get a more accurate asymptotic relation.

Instead of the double summation in the right hand side of (1), in this short note, we will derive a new version of the Stirling formula via an infinite integral as

\[ n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \exp \left(\int_{n}^{\infty} \frac{1}{2} - \frac{x}{x} \, dx\right) \tag{2} \]

and it is applied to give a new and more natural proof of (1).

We will need the following well known Dirichlet test for convergence of infinite integral (see e.g. [5]).

**Lemma.** If \( F(A) = \int_{a}^{A} f(x) \, dx \) is bounded on \([a, \infty)\), \( g(x) \) is monotonic on \([a, \infty)\) and \( \lim_{x \to \infty} g(x) = 0 \), then the infinite integral \( \int_{a}^{\infty} f(x)g(x) \, dx \) is convergence.

### 2. Proof of (2)

From [6] and [7] we may find that

\[ \log n! = (n + \frac{1}{2}) \log n - n + 1 - \int_{1}^{n} \frac{1}{2} - \{x\} \, dx. \tag{3} \]

where \( \{x\} = x - [x] \) and \([x]\) denotes the integral part of \( x \).

Put \( \delta_n := 1 - \int_{1}^{n} \frac{1}{2} - \{x\} \, dx \). By Lemma, it is not difficult to find that the infinite integral

\[ \int_{1}^{\infty} \frac{1}{2} - \{x\} \, dx \]

is convergence, since

\[ F(A) = \int_{1}^{A} \left(\frac{1}{2} - \{x\}\right) \, dx = \int_{[A]}^{A} \left(\frac{1}{2} - \{x\}\right) \, dx = \int_{[A]}^{[A]+\frac{1}{2}} \left(\frac{1}{2} - (A)\right) \, dx = \frac{1}{2}(A)(1 - \{A\}) \]

is bounded on \([1, \infty)\) and \( g(x) = \frac{1}{x} \) is strictly decreasing on \([1, \infty)\) with \( \lim_{x \to \infty} g(x) = 0 \). So, we have
\[
\lim_{n \to \infty} \delta_n = 1 - \int_1^{\infty} \frac{\frac{1}{2} - \{x\}}{x} \, dx := \delta. \tag{4}
\]

Then from (3) we get
\[
n! = \left(\frac{n}{e}\right)^n \sqrt{n} e^{\delta_n}. \tag{5}
\]
Similarly we have
\[
(2n)! = \left(\frac{2n}{e}\right)^{2n} \sqrt{2n} e^{\delta_{2n}}. \tag{6}
\]
By (4) we get
\[
\lim_{n \to \infty} \delta_{2n} = \lim_{n \to \infty} \delta_n = \delta. \tag{7}
\]
Substituting (5), (6) and (7) into the Wallis’ product formula
\[
\lim_{n \to \infty} \frac{(n^!)^2 2^{2n}}{(2n)! \sqrt{n}} = \sqrt{\pi},
\]
we get
\[
e^{\delta} = \sqrt{2\pi}.
\]
Thus (5) may be rewritten in the form
\[
n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi ne} e^{\delta_n - \delta}. \tag{8}
\]
Finally, notice that
\[
\delta_n - \delta = \int_n^{\infty} \frac{\frac{1}{2} - \{x\}}{x} \, dx,
\]
and substitute it into (8), the formula (2) is obtained.

3. A New Proof of (1)

It is immediate to prove (1) by applying (2), since
\[
\int_{n}^{\infty} \frac{1}{x} \frac{x - \{x\}}{x} \, dx = \sum_{k=n}^{\infty} \int_{k}^{k+1} \frac{1}{x} \frac{x - \{x\}}{x} \, dx
\]
\[
= \sum_{k=n}^{\infty} \int_{k}^{k+1} \frac{k+\frac{1}{2} - x}{x} \, dx
\]
\[
= \sum_{k=n}^{\infty} \left[(k + \frac{1}{2}) \log \frac{k+1}{k} - 1\right]
\]
\[
= \sum_{k=n}^{\infty} \left[(1 - \frac{1}{2k} + \frac{1}{3k^2} - \frac{1}{4k^3} + \cdots + (-1)^{j-1} \frac{1}{jk^{j-1}} + \cdots) + \frac{1}{2} \left(\frac{1}{k} - \frac{1}{2k^2} + \frac{1}{3k^3} - \cdots + (-1)^{j-1} \frac{1}{jk^{j-1}} + \cdots\right) - 1\right]
\]
\[
= \sum_{k=n}^{\infty} \sum_{j=2}^{\infty} \frac{j-1}{2j(j+1)} \left(-\frac{1}{k}\right)^j
\]

References


