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A Topology on a Power Set of a Set and Convergence of a Sequence of Sets*

Shozo Niizeki†

Department of Mathematics, Faculty of Science

Kochi University, Kochi 780-8520, Japan

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Abstract

In this paper, first we will introduce a topology into $\boldsymbol{p}(\Omega)$, power set of a set Ω, given by Definition 1.1 and will investigate some properties of this topology. Next we will examine the relation between this topology and the convergence of a sequence of sets in the sense of Definition 1.5. Finally, we will prove that the convergence of a sequence of sets in the sense of Definition 1.5 coincides with its convergence in the sense of this topology. We will formulate this as Theorem 4.3 below.

Keywords and Phrases: *Topological space, Power set of a set, Convergence of a sequence of sets.*

1. Definitions and Notations

In this section, we will give several definitions and notations which will be effectively used in and after Section 2. Throughout this paper, by Ω we mean a fundamental set, that is, all sets treated in this paper are subsets of Ω and by $\boldsymbol{Z}^+$ we mean the set consisting of all positive integers.

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†E-Mail: niizeki@math.Kochi-u.ac.jp

Definition 1.1. By $\boldsymbol{p}(\Omega)$ we mean the set consisting of all subsets of Ω.

Definition 1.2. For $A, B \in \boldsymbol{p}(\Omega)$, we define the symmetric difference $A \triangle B$ by

$$A \triangle B = (A \cap B^c) \cup (A^c \cap B).$$

Definition 1.3. We define subset $\mathscr{F}$ of $\boldsymbol{p}(\Omega)$ by

$$\mathscr{F} = \{F \mid F \text{ is finite subset of } \Omega\}.$$

Definition 1.4. We define the upper limit and the lower limit for a sequence $\{A_n\}_{n=1}^{\infty}$ of sets by

$$\varlimsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i, \quad \varliminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$$

respectively.

Definition 1.5. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets. We say that the sequence $\{A_n\}_{n=1}^{\infty}$ converges to a set A if the following

$$\varlimsup_{n\to\infty} A_n = \varliminf_{n\to\infty} A_n = A$$

holds and we write this as $\lim\limits_{n\to\infty} A_n = A$.

Definition 1.6. For $A \in \boldsymbol{p}(\Omega)$ and $F \in \mathscr{F}$, we define $\mathcal{U}(A|F)$, which is a subset of $p(\Omega)$, by

$$\mathcal{U}(A|F) = \{E \mid (A \triangle E) \cap F = \phi\}.$$

Definition 1.7. For $A \in p(\Omega)$, we define $\boldsymbol{V}(A)$ by

$$\boldsymbol{V}(A) = \{\mathcal{U}(A|F) \mid F \in \mathscr{F}\}.$$

2. Convergence of a Sequence of Sets

In this section, by way of preparation for the following section, we present several. theorems. In order to prove these theorems, we give three lemmas as follows.

Lemma 2.1. *Let both* $\{A_n\}_{n=1}^{\infty}$ *and* $\{B_n\}_{n=1}^{\infty}$ *be decreasing sequences of sets. Then we have*

$$\bigcap_{n=1}^{\infty}(A_n \cup B_n) = \left(\bigcap_{n=1}^{\infty} A_n\right) \cup \left(\bigcap_{n=1}^{\infty} B_n\right).$$

Proof. It is obvious that

$$\bigcap_{n=1}^{\infty}(A_n \cup B_n) \supset \left(\bigcap_{n=1}^{\infty} A_n\right) \cup \left(\bigcap_{n=1}^{\infty} B_n\right). \tag{2.1}$$

Since both $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ are decreasing sequences, for any $m, n \in \boldsymbol{Z}^+$, we have

$$\bigcap_{n=1}^{\infty}(A_n \cup B_n) \subset (A_m \cup B_m) \cap (A_n \cup B_n) \subset A_m \cup B_n.$$

Hence we obtain

$$\bigcap_{n=1}^{\infty}(A_n \cup B_n) \subset \left(\bigcap_{m=1}^{\infty} A_m\right) \cup \left(\bigcap_{n=1}^{\infty} B_n\right). \tag{2.2}$$

Therefore, by (2.1) and (2.2), we complete the proof of Lemma 2.1. □

Corollary 2.2. *Let both* $\{A_n\}_{n=1}^{\infty}$ *and* $\{B_n\}_{n=1}^{\infty}$ *be incresing sequences. Then*

$$\bigcup_{n=1}^{\infty}(A_n \cap B_n) = \left(\bigcup_{n=1}^{\infty} A_n\right) \cap \left(\bigcup_{n=1}^{\infty} B_n\right).$$

Proof. The proof immediately follows from Lemma 2.1 and De Morgan's law. □

Lemma 2.3. *For any sequence* $\{A_n\}_{n=1}^{\infty}$ *of sets,*

$$\varliminf_{n\to\infty} A_n \subset \varlimsup A_n.$$

Proof. For any positive integers m and n, we have

$$\bigcap_{i=m}^{\infty} A_i \subset \bigcup_{i=n}^{\infty} A_i.$$

Hence we obtain

$$\bigcup_{m=1}^{\infty}\bigcap_{i=m}^{\infty} A_i \subset \bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} A_i.$$

By Definition 1.4, this completes the proof of Lemma 2.3. □

Lemma 2.4. *Let* $\{B_n\}_{n=1}^{\infty}$ *be a decreasing sequence of finite sets. If*

$$\bigcap_{n=1}^{\infty} B_n = \phi \tag{2.3}$$

holds true, then there exists an integer $N \in \boldsymbol{Z}^+$ *such that* $B_N = \phi$.

Proof. If $B_k \neq \phi$ for every integer $k \in \boldsymbol{Z}^+$, then by the assumptions for the sequence $\{B_n\}_{n=1}^{\infty}$ we have

$$\bigcap_{n=1}^{\infty} B_n \neq \phi.$$

This contradicts the condition (2.3). Hence the proof of Lemma 2.4 is complete. □

Theorem 2.5. *The following three statements are equivalent* :

(1) $\lim_{n\to\infty} A_n = \phi$;

(2) $\varlimsup_{n\to\infty} A_n = \phi$;

(3) *For any* $F \in \mathscr{F}$, *there exists an integer* $N \in \boldsymbol{Z}^+$ *such that*

$$A_n \cap F = \phi$$

holds true for every integer $n \geq N$.

Proof. First, by Definition 1.5 and Lemma 2.1 we see that the statements (1) and (2) are equivalent. Next, we suppose that (2) holds true and that, for $F \in \mathscr{F}$ we set

$$B_n = \bigcup_{k=n}^{\infty} (A_k \cap F).$$

Hence $\{B_n\}_{n=1}^{\infty}$ is a decreasing sequence, and by (2) we have

$$\bigcap_{n=1}^{\infty} B_n = \left(\overline{\lim}\, A_n\right) \cap F = \phi.$$

Therefore, by Lemma 2.4, $B_N = \phi$ holds true for an integer $N \in \boldsymbol{Z}^+$ and we obtain 3). Finally, we suppose that 3) holds true. Then, for any $F \in \mathscr{F}$, we have

$$\left(\overline{\lim_{n\to\infty}}\, A_n\right) \cap F = \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} (A_k \cap F) \right) = \phi.$$

Hence we obtain 2). Therefore, we complete the proof of Theorem 2.5. □

The following lemma is the most important for proving Theorem 4.1.

Lemma 2.6. *For a set A and a sequence $\{A_n\}_{n=1}^{\infty}$ of sets,*

$$\overline{\lim_{n\to\infty}}\,(A_n \triangle A) = \{(\overline{\lim_{n\to\infty}}\, A_n) \cap A^c\} \cup \{(\underline{\lim_{n\to\infty}}\, A_n)^c \cap A\}.$$

Proof. By Definition 1.2, Lemma 2.1 and Definition 1.4, we obtain

$$\begin{aligned}
\overline{\lim}(A_n \triangle A) &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{(A_k \cap A^c) \cup ({A_k}^c \cap A)\} \\
&= \bigcap_{n=1}^{\infty} \left[\left\{ \left(\bigcup_{k=n}^{\infty} A_k \right) \cap A^c \right\} \cup \left\{ \left(\bigcup_{k=n}^{\infty} {A_k}^c \right) \cap A \right\} \right] \\
&= \left\{ \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) \cap A^c \right\} \cup \left\{ \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} {A_k}^c \right) \cap A \right\} \\
&= \left\{ \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) \cap A^c \right\} \cup \left\{ \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right)^c \cap A \right\} \\
&= \left\{ \left(\overline{\lim_{n\to\infty}}\, A_n \right) \cap A^c \right\} \cup \left\{ \left(\underline{\lim_{n\to\infty}}\, A_n \right)^c \cap A \right\},
\end{aligned}$$

which evidently completes the proof of Theorem 2.6. □

Theorem 2.7. *The following three statements are eqivalent* :

(1) $\lim_{n\to\infty} A_n = A$;

(2) $\lim_{n\to\infty} (A_n \triangle A) = \phi$;

(3) *For any $F \in \mathscr{F}$, there exists a positive number N such that*

$$(A_n \triangle A) \cap F = \phi$$

holds true for any integer $n \geq N$.

Proof. First, if we suppose that the statement (1) holds true, then, by Definition 1.5, Lemma 2.2 and Lemma 2.6, we obtain the statement (2). Next, if we suppose that (2) holds true, then, by Theorem 2.5, we obtain

$$\varlimsup_{n\to\infty} A_n = \varliminf_{n\to\infty} A_n = A.$$

Hence by Definition 1.5 we obtain 1). Therefore 1) and 2) are equivalent. Finally, it follows from Theorem 2.4 that 2) and 3) are equivalent. Consequently, we complete the proof of the Theorem. □

3. Preparatory to the Introduction of a Topology

In this section, we will give several lemmas which are preparatory to the introduction of a topology into $p(\Omega)$.

Lemma 3.1. $\mathcal{U}(A|F)$ *defined by Definition* 1.6 *has following properties* :

1) $A \in \mathcal{U}(A|F)$,

2) $(A \triangle F)^c \in \mathcal{U}(A|F)$.

Proof. The statement (1) follows from Definition 1.6, and the statement (2) follows also from Definition 1.6, since

$$\{(A \triangle F)^c \triangle A\} \cap F = \{(A \triangle F) \triangle A\}^c \cap F = F^c \cap F = \phi$$

holds true. □

Lemma 3.2. *For $A \in p(\Omega)$ and $F \in \mathscr{F}$, if $E_1, E_2 \in \mathcal{U}(A|F)$,*

$$(E_1 \triangle E_2) \cap F = \phi.$$

Proof. We can easily see that

$$\begin{aligned}(E_1 \triangle E_2) \cap F &= \{(E_1 \triangle A) \triangle (E_2 \triangle A)\} \cap F \\ &= \{(E_1 \triangle A) \cap F\} \triangle \{(E_2 \triangle A) \triangle F\} \\ &= \phi \triangle \phi = \phi.\end{aligned}$$

Hence we complete the proof of Lemma 3.2. □

Lemma 3.3. *For a set A and $F_1, F_2 \in \mathscr{F}$, the following statements* (1) *and* (2) *are equivalent* :

(1) $F_1 \subset F_2$;

(2) $\mathcal{U}(A|F_2) \subset \mathcal{U}(A|F_1)$.

Proof. It is obvious from Definition 1.6 that (2) follows from (1). Next, we suppose that (2) holds true. By (2) of Lemma 3.1, we have $(A \triangle F_2)^c \in \mathcal{U}(A|F_2)$ and hence by (2) we have $(A \triangle F_2)^c \in \mathcal{U}(A|F_1)$. Therefore, we obtain

$$\{(A \triangle F_2)^c \triangle A\} \cap F_1 = \phi.$$

Hence we have $F_1 \cap {F_2}^c = \phi$ and we obtain (1). □

Lemma 3.4. *For $A \in \boldsymbol{p}(\Omega)$ and $F_1, F_2 \in \mathscr{F}$,*

$$\mathcal{U}(A|F_1) \cap \mathcal{U}(A|F_2) = \mathcal{U}(A|F_1 \cup F_2).$$

Proof. By Lemma 3.3, we have

$$\mathcal{U}(A|F_1) \cap \mathcal{U}(A|F_2) \supset \mathcal{U}(A|F_1 \cup F_2). \tag{3.1}$$

Next, if $E \in \mathcal{U}(A|F_1) \cap \mathcal{U}(A|F_2)$, then, by Definition 1.5, we have

$$(A \triangle E) \cap F_1 = \phi, \quad (A \triangle E) \cap F_2 = \phi.$$

Hence we obtain $(A \triangle E) \cap (F_1 \cup F_2) = \phi$ and we have $E \in \mathcal{U}(A|F_1 \cup F_2)$. Therefore

$$\mathcal{U}(A|F_1) \cap \mathcal{U}(A|F_2) \subset \mathcal{U}(A|F_1 \cup F_2). \tag{3.2}$$

holds true. By (3.1) and (3.2), we complete the proof of Lemma 3.4. □

Lemma 3.5. *For $A, B \in \boldsymbol{p}(\Omega)$ and $F \in \mathscr{F}$, the following four statements are equivalent:*

(1) $(A \triangle B) \cap F = \phi$;

(2) $A \in \mathcal{U}(B|F)$;

(3) $B \in \mathcal{U}(A|F)$;

(4) $\mathcal{U}(A|F) = \mathcal{U}(B|F)$.

Proof. It is obvious from Definition 1.6 that (2) follows from (1) and that (3) follows from (2). We now suppose that (3) holds true. Then we have $(A \triangle B) \cap F = \phi$, and if $E \in \mathcal{U}(A|F)$, then we have $(A \triangle E) \cap F = \phi$. Therefore, we obtain

$$\begin{aligned}(B \triangle E) \cap F &= \{(A \triangle B) \triangle (A \triangle E)\} \cap F \\ &= \{(A \triangle B) \cap F\} \triangle \{(A \triangle E) \cap F\} \\ &= \phi \triangle \phi = \phi.\end{aligned}$$

Hence we obtain $E \in \mathcal{U}(B|F)$ and we have $\mathcal{U}(A|F) \subset \mathcal{U}(B|F)$. In the same way, we have $\mathcal{U}(B|F) \subset \mathcal{U}(A|F)$. Therefore, we obtain (4). Finally, it is obvious from (1) of Lemma 3.1 and Lemma 3.2 that (1) follows from (4). Consequently, the proof of Lemma 3.5 is complete. $\square$

Theorem 3.6. *Either* (1) *or* (2) *holds true:*

1) $\mathcal{U}(A|F) = \mathcal{U}(B|F)$;

2) $\mathcal{U}(A|F) \cap \mathcal{U}(B|F) = \phi$.

Furthermore, (1) *does not coincide with* (2).

Proof. It is sufficient to show that, if (2) does not hold true, then (1) holds true. We now suppose that $\mathcal{U}(A|F) \cap \mathcal{U}(B|F) \neq \phi$. Then there exists a set E such that

$$E \in \mathcal{U}(A|F) \cap \mathcal{U}(B|F).$$

Hence, by Lemma 3.5, we obtain $(A \triangle E) \cap F = \phi$ and $(B \triangle E) \cap F = \phi$, and we have

$$\begin{aligned}(A \triangle B) \cap F &= \{(A \triangle E) \triangle (B \triangle E)\} \cap F \\ &= \{(A \triangle E) \cap F\} \cap \{(B \triangle E) \cap F\} \\ &= \phi \triangle \phi = \phi.\end{aligned}$$

Therefore, by Lemma 3.5, we have $\mathcal{U}(A|F) = \mathcal{U}(B|F)$ and (1) holds true. By (1) of Lemma 3.1, we have $\mathcal{U}(A|F) \neq \phi$ for any set $F \in \mathscr{F}$. Hence (1) does not coincide with (2). □

Theorem 3.7. *Let $A, B \in \boldsymbol{p}(\Omega)$. Then*

(1) *$\mathcal{U}(A|F) = \mathcal{U}(B|F)$ holds true for any $F \in \mathscr{F}$ if and only if $A = B$;*

(2) *$\mathcal{U}(A|F) \cap \mathcal{U}(B|F) = \phi$ holds true for some $F \in \mathscr{F}$ if and only if $A \neq B$.*

Proof. (1) If $\mathcal{U}(A|F) = \mathcal{U}(B|F)$ holds true for any $F \in \mathscr{F}$, then, by Lemma 3.5, we have $(A \triangle B) \cap F = \phi$ for any $F \in \mathscr{F}$. Hence we have $A \triangle B = \phi$ and we obtain $A = B$. Conversely, if $A = B$ holds true, then it is obvious that

$$\mathcal{U}(A|F) = \mathcal{U}(B|F)$$

holds true for any $F \in \mathscr{F}$.

2) By considering Theorem 3.1 and the contrapositive of (1), we can prove (2). □

4. The Topology and the Convergence of a Sequence of Sets

Under the preparations of the previous section, we can introduce a topology into $\boldsymbol{p}(\Omega)$, that is, we can obtain the following theorem.

Theorem 4.1. *For $A \in \boldsymbol{p}(\Omega)$, $\boldsymbol{V}(A)$ is a fundamental neighbourhood system of A.*

Proof. First, for any $\mathcal{U}(A|F) \in \boldsymbol{V}(A)$, by 1) of Lemma 3.1 we have $A \in \mathcal{U}(A|F)$. Secondly, if $\mathcal{U}(A|F_1)$, $\mathcal{U}(A|F_2) \in \boldsymbol{V}(A)$, then, by Lemma 3.4, we have

$$\mathcal{U}(A|F_1) \cap \mathcal{U}(A|F_2) = \mathcal{U}(A|F_1 \cup F_2) \in \boldsymbol{V}(A).$$

Finally, let $\mathcal{U}(A|F) \in \boldsymbol{V}(A)$. If $B \in \mathcal{U}(A|F)$ then by Lemma 3.5 we have

$$\mathcal{U}(B|F) = \mathcal{U}(A|F). \tag{4.1}$$

and we have $\mathcal{U}(B|F) \in \boldsymbol{V}(A)$. Now, for any $E \in \mathcal{U}(B|F)$, by Lemma 3.5 and (4.1), we have

$$\mathcal{U}(E|F) = \mathcal{U}(A|F), \tag{4.2}$$

and by Lemma 3.5, (4.1) and (4.2), we have $\mathcal{U}(E|F) \in \boldsymbol{V}(B)$. Therefore, we complete the proof of Theorem 4.1. □

By means of Theorem 4.1, we can now introduce a topology $\mathscr{D}$, that is, a system of open sets into $\boldsymbol{p}(\Omega)$ as follows:

$$\mathscr{D} = \{\boldsymbol{O} \mid \text{if } A \in \boldsymbol{O}, \text{ then there exists } \mathcal{U}(A|F) \in \boldsymbol{V}(A) \text{ such that } \mathcal{U}(A|F) \subset \boldsymbol{O}\}.$$

Hence we obtain a topological space $(\boldsymbol{p}(\Omega), \mathscr{D})$. This topolpgical space has the following properties. We now formulate them as two theorems.

Theorem 4.2. *For any $A \in \boldsymbol{p}(\Omega)$, $\boldsymbol{V}(A) \subset \mathscr{D}$.*

Proof. Let $\mathcal{U}(A|F) \in \boldsymbol{V}(A)$. If $B \in \mathcal{U}(A|F)$, then, by Lemma 3.5 and Definition 1.7, we have $\mathcal{U}(B|F) = \mathcal{U}(A|F)$ and have $\mathcal{U}(B|F) \in \boldsymbol{V}(B)$, respectively. Hence we have $\mathcal{U}(B|F) \in \mathscr{D}$ and we obtain $\boldsymbol{V}(A) \subset \mathscr{D}$. □

Theorem 4.3. *The topological space $(\boldsymbol{p}(\Omega), \mathscr{D})$ is a Hausdorff space.*

Proof. The proof of Theorem 4.3 immediately follows from Corollary 3.8 and Theorem 4.1. □

Finally, we examine the relation between the topolpgy introduced by Theorem 4.1 and the convergence of a sequence of sets given by Definition 1.4.

Theorem 4.4. *Let $\{A_n\}_{n=1}^{\infty}$ and A be a sequence of sets and a set in $\boldsymbol{p}(\Omega)$, respectively. Then the following statements* (1) *and* (2) *are equivalent* :

(1) $\lim_{n\to\infty} A_n = A$;

(2) *For any $\mathcal{U}(A|F) \in \boldsymbol{V}(A)$, there exists $N \in \boldsymbol{Z}^+$ such that*

$$A_n \in \mathcal{U}(A|F)$$

holds true for any $n \geq N$.

Proof. The proof of Theorem 4.4 immediately follows from Definition 1.5 and Theorem 2.3. □

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