Relations for Moments of Lower Generalized Order Statistics from Exponentiated Inverted Weibull Distribution

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Abstract

In this paper we consider exponentiated inverted Weibull distribution. Exact expressions and some recurrence relations for single and product moments of lower generalized order statistics are derived. Further the results are deduced for moments of order statistics and lower record values and characterization of this distribution has been considered on using conditional expectation of function of lower generalized order statistics and a recurrence relation for single moments. The first four moments, mean and variance of order statistics and record values are computed for various values of parameters.

Keywords and Phrases: Lower generalized order statistics, Order statistics, Lower record values, Exponentiated inverted Weibull distribution, Single and product moments, Recurrence relations and characterization.
1. Introduction

Kamps [14] introduced the concept of generalized order statistics (gos). It is known that ordinary order statistics, sequential order statistics, Stigler’s order statistics and upper record values are special cases of gos. In this article we will consider the lower generalized order statistics (l gos) defined as follows:

Let \( n \in \mathbb{N}, \ k \geq 1, \ m \in \mathbb{R}, \) be the parameters such that
\[
\gamma_r = k + (n-r)(m+1) > 0 \quad \text{for all} \ 1 \leq r \leq n.
\]

Then \( X^*(1,n,m,k), \ldots, X^*(n,n,m,k) \) are called l gos from an absolutely continuous distribution function \((df) \ F(x)\) with the probability distribution function \((pdf) \ f(x)\) if their joint pdf has the form
\[
f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} [F(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)) \tag{1.1}
\]
for \( F^{-1}(l) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0). \)

The marginal pdf of \( r \)-th l gos, \( X^*(r,n,m,k), \) is
\[
f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} [F(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)) \tag{1.2}
\]
and the joint pdf of \( X^*(r,n,m,k) \) and \( X^*(s,n,m,k), \) \( 1 \leq r < s \leq n, \) is
\[
f_{X^*(r,n,m,k),X^*(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(s-1)!} [F(x)]^{m} f(x) g_m^{r-1}(F(x)) \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_{s-1}} f(y), \ x > y, \tag{1.3}
\]
where
\[
C_{r-1} = \prod_{i=1}^{r} \gamma_i,
\]
\[
h_m(x) = \begin{cases} 
- \frac{1}{m+1} x^{m+1}, & m \neq -1 \\
- \ln x, & m = -1
\end{cases}
\]
and
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\[ g_m(x) = h_m(x) - h_n(1), \quad x \in [0, 1). \]

We shall also take \( X^*(0,n,m,k)=0 \). If \( m=0, k=1 \), then \( X^*(r,n,m,k) \) reduces to the \((n-r+1)-\)th order statistic, \( X_{n-r+1:n} \) from the sample \( X_1, X_2, \ldots, X_n \) and when \( m=-1 \), then \( X^*(r,n,m,k) \) reduces to the \( r-\)th lower \( k \) record value (Pawlas and Szynal [9]). The work of Burkschat et al. [8] may also refer for \( lgos \).

Recurrence relations for single and product moments of \( lgos \) from the inverse Weibull distribution are derived by Pawlas and Syznal [9]. Ahsanullah [7] and Mbah and Ahsanullah [2] characterized the uniform and power function distributions based on distributional properties of \( lgos \), respectively. Khan and Kumar [10, 11 & 12], Kumar and Khan [3] and Kumar [4, 5] have established recurrence relations for moments of \( lgos \) from exponentiated Pareto, exponentiated gamma and generalized exponential, exponentiated log-logistic, exponentiated Kumaraswamy and J-shaped distributions.

In this study we obtain some recurrence relations for single and product moments of \( lgos \) from exponentiated inverted Weibull distribution. The relations for order statistics and lower records are deduced from the relations derived. Further, the distribution has been characterized on using conditional expectation of function of \( lgos \) and a recurrence relation for single moment of \( lgos \).

A random variable \( X \) is said to have exponentiated inverted Weibull distribution (Flaih et al., [1]) if its pdf is of the form

\[ f(x) = \alpha \beta x^{-(\beta+1)} e^{-\alpha x^{-\beta}}, \quad x > 0, \quad \alpha, \beta > 0 \]  

(1.4)

and the corresponding df is

\[ F(x) = e^{-\alpha x^{-\beta}}, \quad x > 0, \quad \alpha, \beta > 0. \]  

(1.5)

It is easy to see that

\[ x^{\beta+1} f(x) = \alpha \beta F(x) \]  

(1.6)

The inverted Weibull and inverted exponential distributions are considered as special cases of the exponentiated inverted Weibull distribution when \( \alpha = 1 \) and
\( \beta = 1, \alpha = 1 \), respectively. More details on this distribution can be found in Flaih et al., [1].

2. Relations for single moments

In this section, we have derived the exact expressions for \( lgos \) from exponentiated inverted Weibull distribution. Further these results are used to evaluate the first four moments, mean and variance of order statistics and record values are presented in table 2.1, 2.2, 2.3 and table 2.4. We shall first establish the exact expression for \( E[X^*j(r,n,m,k)] \). Using (1.2), we have when \( m \neq -1 \)

\[
E[X^*j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [F(x)]^{\gamma_r-1} f(x) g_m^{-1}(F(x))dx
\]

\[
= \frac{C_{r-1}}{(r-1)!} I_j(\gamma_r - 1, r - 1), \tag{2.1}
\]

where

\[
I_j(a,b) = \int_0^\infty x^j [F(x)]^a f(x) g_m^b(F(x))dx.
\] (2.2)

On expanding \( g_m^b(F(x)) = \left[ \frac{1}{m+1} [1 - (F(x))^{m+1}] \right]^b \) binomially in (2.2), we get when \( m \neq -1 \)

\[
I_j(a,b) = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} \int_0^\infty x^j [F(x)]^{u+m+1} f(x)dx. \tag{2.3}
\]

Making the substitution \( t = -\ln F(x) \) in (2.3), we find that

\[
I_j(a,b) = \frac{\alpha^j \beta}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} \int_0^\infty t^{-j/\beta} e^{-[a+u(m+1)+1] t} dt
\]
\[ E[X^*j(r,n,m,k)] = \frac{\alpha^{j/\beta} C_{r-1}}{(r-1)!} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{\Gamma(1-j/\beta)}{(r-u)(\gamma_{r-u})^{1-j/\beta}}, \quad \beta > j, \text{ and } j = 0,1,2,\ldots \] (2.5)

and when \( m = -1 \) that

\[ E[X^*j(r,n,m,k)] = 0 \quad \text{as} \quad \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} = 0. \]

Since (2.5) is of the form \( \frac{0}{0} \) at \( m = -1 \), therefore, we have

\[ E[X^*j(r,n,m,k)] = A \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{[k + (n-r+u)(m+1)]^{-(1-j/\beta)}}{(m+1)^{r-1}}, \] (2.6)

where

\[ A = \frac{\alpha^{j/\beta} C_{r-1}}{(r-1)!} \Gamma(1-j/\beta). \]

Differentiating numerator and denominator of (2.6) \((r-1)\) times with respect to \( m \), we get

\[ E[X^*j(r,n,m,k)] = A \sum_{u=0}^{r-1} (-1)^u (r-1)^u \binom{r-1}{u} \frac{[1-j/\beta](2-j/\beta)\ldots(r-1-j/\beta)(n-r+u)^{-1}}{(r-1)! [k + (n-r+u)(m+1)]^{r-j/\beta}} \]
\[ = A \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{(1- j/ \beta)(2-j/ \beta)\ldots(r-1-j/ \beta)(r-n-u)^{r-1}}{(r-1)! [k+(n-r+u)(m+1)]^{r-j/ \beta}}. \]

On applying L’ Hospital rule, we have

\[
\lim_{m \to -1} E[X^* j(r,n,m,k)] = A \frac{(1-j/ \beta)(2-j/ \beta)\ldots(r-1-j/ \beta)}{(r-1)! k^{r-j/ \beta}} \times \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} (r-n-u)^{r-1}. \tag{2.7}
\]

But for all integers \( n \geq 0 \) and for all real numbers \( x \), we have Ruiz [13]

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)^n = n!. \tag{2.8}
\]

Now substituting (2.8) in (2.7) and simplifying, we find that

\[
E[X^* j(r,n,-1,k)] = E[(X_{L(r)}^j)^k] = \frac{(\alpha k)^{j/ \beta}}{(r-1)!} \Gamma(r-j/ \beta). \tag{2.9}
\]

**Special cases**

i) Putting \( m=0 \), \( k=1 \) in (2.5), the exact expression for the single moments of order statistics of the exponentiated inverted Weibull distribution can be obtained as

\[
E[X^j_{n-r+1,n}] = \alpha^{j/ \beta} C_{rn} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{\Gamma(1-j/ \beta)}{(n-r+1+u)^{1-j/ \beta}}. \]

That is
Relations for Moments of Lower Generalized Order

\[ E[X^r_{rn}] = \alpha^{j/\beta} C_{rn} \sum_{u=0}^{n-r} (-1)^u \binom{n-r}{u} (r+u) \Gamma(1+j/\beta) \frac{(1-j/\beta)}{(r+u)^{1+j/\beta}}, \]

where

\[ C_{rn} = \frac{n!}{(r-1)!(n-r)!}. \]

ii) Putting \( k = 1 \) in (2.9), we get the exact expression for the single moments of lower records for the exponentiated inverted Weibull distribution can be obtained as

\[ E[X^{*j}(r,n,-1,1)] = E[X^j_{L(r)}] = \frac{\alpha^{j/\beta}}{(r-1)!} \Gamma(r-j/\beta). \]

Recurrence relations for single moments of \( l \)gos from (1.6) can be obtained in the following theorem.

**Theorem 2.1** For the distribution given in (1.4) and for \( 2 \leq r \leq n, \ n \geq 2 \) and \( k = 1,2,\ldots, \)

\[ E[X^{*j+\beta+1}(r,n,m,k)] = \frac{\alpha \beta \gamma_r}{j+1} \left( E[X^{*j+1}(r-1,n,m,k)] - E[X^{*j+1}(r,n,m,k)] \right). \]

**Proof** From (1.2) and (1.6), we have

\[ E[X^{*j+\beta+1}(r,n,m,k)] = \frac{\alpha \beta C_{r-1}}{(r-1)!} \int_0^{\infty} x^{j} [F(x)]^{\gamma_r} g_{m-1}^{r-1} (F(x)) dx. \]

Integrating by parts treating \( x^j \) for integration and the rest of the integrand for differentiation, we get

\[ E[X^{*j+\beta+1}(r,n,m,k)] = \frac{\alpha \beta \gamma_r}{j+1} \left[ \frac{C_{r-2}}{(r-2)!} \int_0^{\infty} x^{j+1} [F(x)]^{\gamma_r-1} f(x) g_{m-2}^{r-2} (F(x)) dx \right]. \]
and hence the result.

**Remark 2.1** Setting \( m = 0, k = 1 \) in (2.10), we obtain a recurrence relation for single moments of order statistics of the exponentiated inverted Weibull distribution in the form

\[
E[X_{n-r+ln}^{j+\beta+1}] = \frac{\alpha \beta (n-r+1)}{j+1} \{E[X_{n-r+2ln}^{j+1}] - E[X_{n-r+ln}^{j+1}]\}.
\]

That is

\[
E[X_{ln}^{j+1}] = E[X_{r-1n}^{j+1}] + \frac{j+1}{\alpha \beta (r-1)} E[X_{r-1n}^{j+\beta+1}].
\]

**Remark 2.2** Putting \( m = -1 \), in Theorem 2.1, we get a recurrence relation for single moments of lower \( k \) record values from exponentiated inverted Weibull distribution in the form

\[
E[(X_{L(r)}^{j+\beta+1}})^k] = \frac{\alpha \beta k}{j+1} \{E[(X_{L(r-1)}^{j+1})^k] - E[(X_{L(r)}^{j+1})^k]\}.
\]

**Remark 2.3** Putting \( \alpha = 1 \), in (2.10), we get the recurrence relation for single moments of lower \( k \) record values from exponentiated inverted Weibull distribution.

**Remark 2.4** For \( \alpha = 1 \) and \( \beta = 1 \), the relation in (2.10), gives the recurrence relation for single moments of lower \( k \) record values from inverted exponential distribution.

**Table 2.1:** First four moments of order statistics

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**Table 2.2** Variance of order statistics

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Table 2.3  First four moments of lower records

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Table 2.4  Variance of lower records

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<th>(\beta = 5)</th>
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3. Relations for product moments

**Theorem 3.1**  For the given exponentiated inverted Weibull distribution in (1.3) and $n \geq 2$, $m \in \mathbb{R}$, $1 \leq r < r+1 \leq n$,

\[
E[X^{ni}(r,n,m,k)X^{j+i+1}(r+1,n,m,k)] = \frac{\alpha \beta \gamma}{j+1} E[X^{ni+j+i+1}(r,n,m,k)]
\]

- \[E[X^{ni}(r,n,m,k)X^{j+1}(r+1,n,m,k)] \right]

and for $1 \leq r < s \leq n$, $s-r \geq 2$ and $i, j \geq 0$,

\[
E[X^{ni}(r,n,m,k)X^{j+i+1}(s,n,m,k)]
\]

\[= \frac{\alpha \beta \gamma_s}{j+1} E[X^{ni}(r,n,m,k)X^{j+i+1}(s-1,n,m,k)]
\]

- \[E[X^{ni}(r,n,m,k)X^{j+1}(s,n,m,k)] \right].

**Proof**  From (1.3), we have

\[
E[X^{ni}(r,n,m,k)X^{j+i+1}(s,n,m,k)]
\]

\[= \frac{C_{s-1}}{(s-r-1)!} \int_0^\infty \int_0^\infty x^j [F(x)]^m g_m r^{-1}(F(x))I(x)dx,
\]

where

\[
I(x) = \int_0^\infty y^{j+i+1} [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s-1} f(y)dy
\]
Relations for Moments of Lower Generalized Order

\[
= \alpha \beta \int_0^x y^j [h_m(F(y)) - h_m(F(x))]^{s-r} y^r dy
\]

upon using the relation in (1.6). Integrating now by parts treating \(y^j\) for integration and the rest of the integrand for differentiation, we obtain when \(s = r + 1\) that

\[
I(x) = \frac{\alpha \beta \gamma_{r+1}}{j+1} \left\{ \frac{x^{j+1}}{\gamma_{r+1}} [F(x)]^{\gamma_{r+1}} - \int_0^x y^{j+1} [F(y)]^{\gamma_{r+1} - 1} f(y) dy \right\}
\]

and when \(s > r + 1\) that

\[
I(x) = \frac{\alpha \beta \gamma_s}{j+1} \left\{ \frac{(s-r-1)}{\gamma_s} \int_0^x y^{j+1} [h_m(F(y)) - h_m(F(x))]^{s-r-2} [F(y)]^{\gamma_s + m} f(y) dy - \int_0^x y^{j+1} [h_m(F(y)) - h_m(F(x))]^{s-r-1} f(y) dy \right\}.
\]

Upon substituting the above expressions for \(I(x)\) in (3.3), we have, after simplifications, the recurrence relations (3.1) and (3.2).

**Remark 3.1** Setting \(m = 0\), \(k = 1\), in (3.2), we obtain recurrence relations for product moments of order statistics for the exponentiated inverted Weibull distribution in the form

\[
E[X_{n-r+1:n}^i X_{n-s+1:n}^{j+\beta+1}] = \frac{\alpha \beta (n-s+1)}{j+1} \left\{ E[X_{n-r+1:n}^i X_{n-s+2:n}^{j+1}] - E[X_{n-r+1:n}^i X_{n-s+1:n}^{j+1}] \right\}.
\]

That is

\[
E[X_{r:n}^{i+j+1} X_{s:n}^j] = E[X_{r-l:n}^{i+j} X_{s:n}^j] + \frac{i+1}{\alpha \beta (r-1)} E[X_{r-l:n}^{i+\beta+1} X_{s:n}^j].
\]
Remark 3.2 Setting \( m = -1 \) and \( k \geq 1 \), in Theorem 3.1, we get the recurrence relations for product moments of lower \( k \) record values from exponentiated inverted Weibull distribution in the form

\[
E[(X_{L(r)}^i)^k (X_{L(s)}^{j+1})^k] = \frac{\alpha \beta k}{j+1} \{E[(X_{L(r)}^i)^k (X_{L(s-1)}^{j+1})^k] - E[(X_{L(r)}^i)^k (X_{L(s)}^{j+1})^k]\}.
\]

Remark 3.3 Putting \( \alpha = 1 \), in (3.1) and (3.2), we deduce the recurrence relations for product moments of lower \( s \) from inverted Weibull distribution.

Remark 3.4 For \( \theta = 1 \) and \( \beta = 1 \), the relations in (3.1) and (3.2) give the recurrence relations for product moments of lower \( s \) from inverted exponential distribution.

Remark 3.5 At \( s = 0 \), Theorem 3.1 can be reduced to Theorem 2.1.

4. Characterization

This Section, contains characterization of exponentiated inverted Weibull distribution by using conditional expectation of function of \( l \) gos and a recurrence relation for single moments of \( l \) gos.

Let \( X^*(r,n,m,k) \), \( r=1,2,\ldots,n \) be \( l \) gos from a continuous population with \( df \) \( F(x) \) and \( pdf \) \( f(x) \), then the conditional \( pdf \) of \( X^*(s,n,m,k) \) given \( X^*(r,n,m,k) = x \), \( 1 \leq r < s \leq n \), in view of (1.2) and (1.3), is

\[
 f_{X^*(s,n,m,k)|X^*(r,n,m,k)}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}} [F(x)]^{m-y_r+1} \\
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{y_s-1} f(y). \tag{4.1}
\]

Theorem 4.1 Let \( X \) be a non-negative random variable having an absolutely continuous distribution function \( F(x) \) with \( F(0) = 0 \) and \( 0 < F(x) < 1 \) for all \( x > 0 \), then
Relations for Moments of Lower Generalized Order

\[ E[\xi(X^*(s,n,m,k)) \mid X^*(l,n,m,k) = x] = e^{-\alpha x^{-\beta}} \prod_{j=1}^{s-1} \left( \frac{\gamma_{l+j}}{\gamma_{l+j} + 1} \right), \quad l = r, r + 1 \quad (4.2) \]

if and only if

\[ F(x) = e^{-\alpha x^{-\beta}}, \quad x > 0, \quad \alpha, \beta > 0, \]

where

\[ \xi(x) = e^{-\alpha y^{-\beta}}. \]

**Proof**  From (4.1), we have

\[ E[\xi(X^*(s,n,m,k)) \mid X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \]

\[ \times \int_0^x e^{-\alpha \gamma^{-\beta}} \left[ 1 - \left( \frac{F(y)}{F(x)} \right)^{m+1} \right]^{s-r-1} \left( \frac{F(y)}{F(x)} \right)^{\gamma_{s-1}} \frac{f(y)}{F(x)} \, dy. \quad (4.3) \]

By setting \( u = \frac{F(y)}{F(x)} e^{-\alpha \gamma^{-\beta}} \) from (1.4) in (4.3), we obtain

\[ E[\xi(X^*(s,n,m,k)) \mid X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \]

\[ \times e^{-\alpha x^{-\beta}} \int_0^1 u^{\gamma_{s-1}} (1-u^{m+1})^{s-r-1} \, du. \quad (4.4) \]

Again by setting \( t = u^{m+1} \) in (4.4), we get

\[ E[\xi(X^*(s,n,m,k)) \mid X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r}} \]
\begin{equation}
\times e^{-\alpha x^{-\beta}} \int_0^{k+1 \over m+1} t^{n-s-1} (1-t)^{s-r-1} dt
\end{equation}

\begin{align*}
C_{s-1} e^{-\alpha x^{-\beta}} & \frac{\Gamma\left(\frac{k+1}{m+1} + n - s\right)}{C_{r-1} (m+1)^{s-r}} \\
C_{r-1} (m+1)^{s-r} & \frac{\Gamma\left(\frac{k+1}{m+1} + n - r\right)}{\Gamma\left(\frac{k+1}{m+1} + n - r\right)} \\
= & e^{-\alpha x^{-\beta}} \prod_{j=1}^{s-r} \left(\frac{\gamma_{r+j}}{\gamma_{r+j} + 1}\right)
\end{align*}

and hence the relation in (4.2).

To prove sufficient part, we have from (4.1) and (4.2)

\begin{equation}
\frac{C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r-1}} \int_0^{x} e^{-\alpha y^{-\beta}} \left[(F(x))^{m+1} - (F(y))^{m+1}\right]^{s-r-1}
\times [F(y)]^{\gamma_{s-1}} f(y) dy = [F(x)]^{\gamma_{r+1}} H_r(x),
\end{equation}

where

\begin{equation}
H_r(x) = e^{-\alpha x^{-\beta}} \prod_{j=1}^{s-r} \left(\frac{\gamma_{r+j}}{\gamma_{r+j} + 1}\right).
\end{equation}

Differentiating (4.5) both the sides with respect to \( x \), we get

\begin{equation}
\frac{C_{s-1} [F(x)]^m f(x)}{(s-r-2)! C_{r-1} (m+1)^{r-r-2}} \int_0^{x} e^{-\alpha y^{-\beta}} \left[(F(x))^{m+1} - (F(y))^{m+1}\right]^{r-r-2}
\times [F(y)]^{\gamma_{s-1}} f(y) dy = H'_r(x) [F(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x) [F(x)]^{\gamma_{r+1}-1} f(x)
\end{equation}

or
\[
\gamma_{r+1} H_{r+1}(x) [F(x)]^{\gamma_{r+2} + m} f(x)
= H'_r(x) [F(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x) [F(x)]^{\gamma_{r+1} - 1} f(x),
\]
where
\[
H'_r(x) = \alpha \beta x^{-(\beta+1)} e^{-\alpha x^{-\beta}} \prod_{j=1}^{s-r} \left( \frac{\gamma_{r+j}}{\gamma_{r+j} + 1} \right),
\]
\[
H_{r+1}(x) - H_r(x) = e^{-\alpha x^{-\beta}} \prod_{j=1}^{s-r} \left( \frac{\gamma_{r+j}}{\gamma_{r+j} + 1} \right).
\]
Therefore,
\[
\frac{f(x)}{F(x)} = \frac{H'_r(x)}{\gamma_{r+1} [H_{r+1}(x) - H_r(x)]} = \alpha \beta x^{-(\beta+1)}
\]
which proves that
\[
F(x) = e^{-\alpha x^{-\beta}}, \quad x > 0, \quad \alpha, \beta > 0.
\]

**Theorem 4.2** Let \( X \) be a non-negative random variable having an absolutely continuous distribution function \( F(x) \) with \( F(0) = 0 \) and \( 0 < F(x) < 1 \) for all \( x > 0 \), then
\[
E[X^{*j+\beta+1}(r, n, m, k)] = \frac{\alpha \beta \gamma_r}{j+1} [E[X^{*j+1}(r-1, n, m, k)] - E[X^{*j+1}(r, n, m, k)]] \quad (4.6)
\]
if and only if
\[
F(x) = e^{-\theta(x) \beta}, \quad x > 0, \quad \alpha, \beta > 0.
\]
Proof The necessary part follows immediately from equation (2.10). On the other hand if the recurrence relation in equation (4.6) is satisfied, then on using equation (1.2), we have

\[
\frac{C_{r-1}}{(r-1)!} \int_{0}^{\infty} x^{j+\beta+1} [F(x)]^{\gamma_r-1} f(x) g_m^{r-1} (F(x)) dx
\]

\[= \frac{\alpha \beta (r-1)}{j+1} \frac{C_{r-1}}{(r-1)!} \int_{0}^{\infty} x^{j+1} [F(x)]^{\gamma_r+m} f(x) g_m^{r-2} (F(x)) dx
\]

\[- \frac{\alpha \beta \gamma_r}{j+1} \frac{C_{r-1}}{(r-1)!} \int_{0}^{\infty} x^{j+1} [F(x)]^{\gamma_r-1} f(x) g_m^{r-1} (F(x)) dx
\]

\[= \frac{\alpha \beta}{j+1} \frac{C_{r-1}}{(r-1)!} \int_{0}^{\infty} x^{j+1} [F(x)]^{\gamma_r} f(x) g_m^{r-2} (F(x))
\]

\[\times \left\{ (r-1)[F(x)]^m - \frac{\gamma_r g_m (F(x))}{F(x)} \right\} dx.
\]

(4.7)

Let \( h(x) = -[F(x)]^{\gamma_r} g_m^{r-1} (F(x)) \) (4.8)

and \( h'(x) = [F(x)]^{\gamma_r} f(x) g_m^{r-2} (F(x)) \left\{ (r-1)[F(x)]^m - \frac{\gamma_r g_m (F(x))}{F(x)} \right\} \).

Thus,

\[
\frac{C_{r-1}}{(r-1)!} \int_{0}^{\infty} x^{j+\beta+1} [F(x)]^{\gamma_r-1} f(x) g_m^{r-1} (F(x)) dx
\]

\[= \frac{\alpha \beta}{j+1} \frac{C_{r-1}}{(r-1)!} \int_{0}^{\infty} x^{j+1} h'(x) dx.
\]

(4.9)

Now integrating RHS in (4.9) by parts and using the value of \( h(x) \) from (4.8), we get

\[
\frac{C_{r-1}}{(r-1)!} \int_{0}^{\infty} x^{j+\beta+1} [F(x)]^{\gamma_r-1} f(x) g_m^{r-1} (F(x)) dx = \alpha \beta \frac{C_{r-1}}{(r-1)!} \int_{0}^{\infty} x^{j} [F(x)]^{\gamma_r} g_m^{r-1} (F(x)) dx
\]

which reduces to
\[
\frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [F(x)]^{r-1} g_m^{-1}(F(x))(\alpha \beta F(x) - x^\beta f(x)) \, dx = 0. \tag{4.10}
\]

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, [6]) to equation (4.10), we get

\[
\frac{f(x)}{F(x)} = \alpha \beta x^{-(\beta+1)}
\]

which proves that

\[
F(x) = e^{-\alpha x^{-\beta}}, \quad x > 0, \quad \alpha, \beta > 0.
\]

**Computations of the moments and their Applications**

The exact and explicit expressions for the single moments of order statistics and record statistics allow us to evaluate the mean and variance of all order statistics and record statistics. Table 2.1, 2.2 and Table 2.3, 2.4 present the mean and variance of order statistics and record statistics of exponentiated inverted Weibull distribution respectively.

**5. Conclusion**

This paper deals with the $lgos$ from the exponentiated inverted Weibull distribution. Some explicit expressions and recurrence relations for single and product moments are derived. Characterizing results of this distribution has been obtained by using conditional expectation of function of $lgos$ and a recurrence relation for single moments of $lgos$. Special cases are also deduced.

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**References**


